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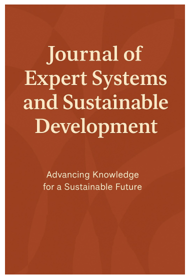

Raptean Fuzzy Set: A Fuzzy Extension and Its Application in Sustainable Evaluation of Clean Energy Barriers

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ABSTRACT

This study introduces the concept of the Raptean fuzzy set (RFS), characterized by a three-dimensional (3-D) fuzzy membership grade, representing a sophisticated expansion of the picture fuzzy set (PiFS), spherical fuzzy set (SFS), and Fermatean fuzzy set (FFS). Subsequently, algebraic operations on RFS, incorporating fundamental properties, are considered for better handling of uncertainty, inspired by the preliminary concept of fuzzy sets. Additionally, the score and accuracy functions have been proposed to defuzzify Raptean fuzzy numbers (RFNs) for the objective of ranking. Moreover, this study presents aggregation operators (AOs) for RFS, utilized for the aggregation of RF information, and provides rigorous theorems along with their corresponding proofs. Furthermore, the Euclidean distance and the Zhang and Xu distance have been introduced for two RFSs. Subsequently, we develop a Raptean fuzzy Weighted Aggregated Sum Product Assessment (WASPAS) method and an innovative ranking technique utilizing the Raptean fuzzy distance measure to tackle the complexities associated with multi-criteria group decision-making (MCGDM). Additionally, this study presents an application of MCGDM and addresses the sustainable evaluation of clean energy barriers, thereby demonstrating the practicability of the introduced models. Finally, the examined comparative and sensitivity analyses validate the stability and efficiency of the suggested MCGDM models and verify the greater decision space for RFS in decision-making problems.

1. Introduction

Extension of fuzzy sets to decision-making situations requiring multiple factors and subjective assessments, optimal decisions are challenging due to inadequate information and incomplete expert knowledge. To address uncertainties, Zadeh introduced fuzzy sets (FS) in 1965 [1], by involving membership grade (MG) with values ranging from 0 to 1. Recognising the need to consider non-membership grade (NMG), Atanassov [2] introduced the intuitionistic fuzzy set (IFS). However, IFS has limitations, as a combination of MG and NMG may surpass 1. To fix this, Yager [3] stated the Pythagorean fuzzy set (PFS), ensuring combine the squares of MG and NMG is between 0 and 1. A general class of IFS and PFS known as q-rung orthopair fuzzy sets (q-ROFS) was introduced by Yager [4]. In these sets, the sum of the qth powers of the MG and NMG is bounded by one. Later, Cuong [5] pro-

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posed the picture fuzzy set (PiFS) as three-dimensional (3-D) membership functions, such as positive membership or membership grade, negative membership grade or non-membership grade and neutral or hesitancy grade (HG), such that the sum of all three grades lies between 0 and 1. In PiFS another grade can be determined by subtracting the total of MG, NMG, and HG from 1, which is referred to as the refusal degree. However, the total of all three grades may be more than 1 in many ambiguous situations, including decision-making issues. These are leading to the development of spherical fuzzy set (SFS) by Kahraman and Gündoğdu [6]. SFS ensures the square total of three grades stays between 0 and 1. Building on these concepts, Gündoğdu and Kahraman [7] established interval-valued spherical fuzzy set (IVSFS) as a continuation of interval-valued fuzzy set (IVFS) [8], interval-valued intuitionistic fuzzy set (IVIFS) [9], interval-valued type-2 fuzzy set (IVT2FS) [10, 11], interval-valued Pythagorean fuzzy set (IVPFS) [12] and SFS [6]. Senapati and Yager [13] later developed the idea of Fermateran fuzzy set (FFS), which is a greater generalization of IFS and PFS in that the cube sum of MG and NMG should be between 0 and 1. In order to obtain a more thorough understanding, Jeevaraj [14] expanded the idea of FFS to include interval-valued Fermatean fuzzy set (IVFF), which is produced by the intervals of lower and upper MG and NMG.

In the meantime, IFS has been effectively used in multi-criteria decision making (MCDM) by introducing averaging, geometric, and order-weighted aggregation operators [15–17]. Yager [18] extends Pythagorean fuzzy membership grades (PFMGs) with MCDM problems. Yager and Abbasov [19] extend the MCDM with PFMGs and complex numbers. Further, Peng and Yang [20] studied some more properties for aggregating the Pythagorean fuzzy information. Garg [21, 22] extended the direction of PFS with MCDM applications. Afterwards, SFS was introduced by Kahraman and Gündoğdu [6], and FFS by Senapati and Yager [13], advancing the research in the decision-making field, as respectively 3-D and 2-D membership functions of these sets make a larger domain for the decision experts. Spherical fuzzy Weighted Aggregated Sum Product ASsessment (SF-WASPAS) for industrial robot selection [23], SF Technique for Order Preference by the Similarity to Ideal Solution (SF-TOPSIS) for air-condition supplier selection problem [24], SF Analytic Hierarchy Process (SF-AHP) for renewable energy location selection [25] and others' hybrid SF multi-criteria group decision-making (MCGDM) models [26–30]. FF weighted averaging and geometric operators for tackling the MCDM application [31], FF TOPSIS for evaluating occupational risk in manufacturing [32], FF multi-objective optimization based on the ratio analysis with the full multiplicative form (MULTIMOORA) method for choosing a charging station for electric vehicles [33], FF hybridizes model with MEthod based on the REmoval effects of Criteria (MEREC) and Combined Compromise Solution (CoCoSo) method for urban transport planning in COVID-19 pandemic [34], Interval-valued FF WASPAS method for e-waste recycling [35], and others' hybrid FF MCGDM models [36–38].

Highly motivated by the foundation of spherical fuzzy sets and Fermatean fuzzy sets, and the applicability of these sets in MCDM applications, we have proposed a new extension of fuzzy sets called the Raptean fuzzy set (RFS) with 3-D membership grades, which is a direct extension of picture fuzzy sets, spherical fuzzy sets and Fermatean fuzzy sets. The key contributions of this current study are as follows:

- Introduced the concept of RFS and its fundamental algebraic operations.
- Proposed the score function and accuracy function, studied the characteristics of the score function depending on membership grades, and defined the comparison procedure for RFNs with diverse examples.
- Proposed two RF-AOs, namely "Raptean fuzzy weighted arithmetic mean (RFWAM) operator" and "Raptean fuzzy weighted geometric mean (RFWGM) operator" based on algebraic operations of RFSs, and also studied the fundamental properties of these two AOs.

- Defined the distance measure of RFSs by Euclidean distance [39] and, Zhang and Xu distance [40], and studied their characteristics.
- Proposed two MCGDM frameworks under Raptean fuzzy environments, one according to the distance measure technique and another is the Raptean fuzzy WASPAS method, and, to incorporate the Raptean fuzzy information, defined a nine-point linguistic variable scale satisfied by the condition of RFSs.
- Illustrate an application on MCGDM based problem namely "clean energy barriers evaluation" [41]. Further, compared the results with the proposed two Raptean fuzzy MCGDM frameworks and examined the sensitivity of the obtained results.

The gist of the paper is stated as follows: Section 2 introduced the concept of RFS and the membership space of this set. Section 3 defined the algebraic operations and fundamental properties of RFSs. Section 4 presented AOs to aggregate the Raptean fuzzy information and defined the Euclidean distance and Zhang and Xu distance for the distance measure of RFNs. Section 5 presented two MCGDM frameworks based on the Raptean fuzzy distance measure and the Raptean fuzzy WASPAS method. Section 6 illustrated an example of an MCGDM-based problem, namely "clean energy barriers evaluation", analyzed with sensitivity and comparative study. Section 7 concludes this research and outlines future studies.

2. Raptean Fuzzy Set

This section presents the definition of the proposed Raptean fuzzy set and discusses the space of membership grades (MGs) of IFS, PFS, FFS, PiFS, SFS, and RPSs.

Definition 2.1. *Let U be a universal set, and a Raptean fuzzy set (RFS) $\tilde{R}$ on U be defined as*

$$\tilde{R} = \{(x, \langle \alpha_{\tilde{R}}(x), \beta_{\tilde{R}}(x), \gamma_{\tilde{R}}(x)\rangle) | \forall x \in U\} \tag{1}$$

where, $\alpha_{\tilde{R}}(x), \beta_{\tilde{R}}(x), \gamma_{\tilde{R}}(x) \in [0,1], \forall x \in U$ are called membership grade (MG), non-membership grade (NMG) and hesitancy grade (HG) of x to $\tilde{R}$ respectively and satisfy the condition given below:

$$0 \leq \alpha^3_{\tilde{R}}(x) + \beta^3_{\tilde{R}}(x) + \gamma^3_{\tilde{R}}(x) \leq 1, \forall x \in U. \tag{2}$$

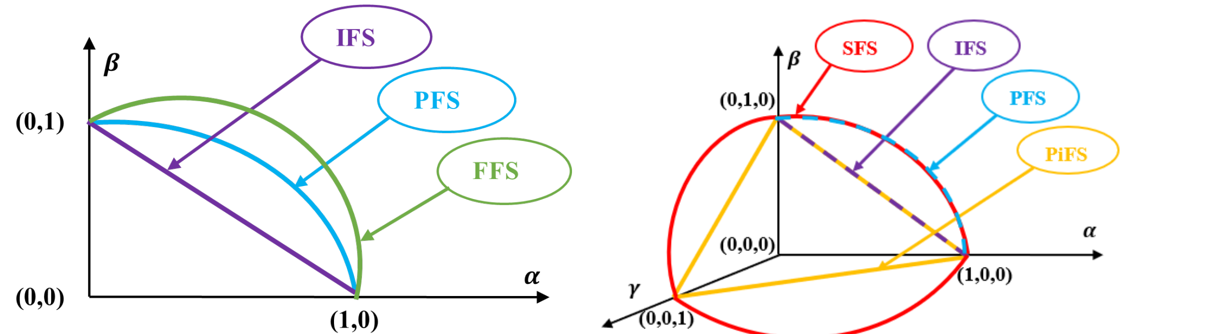
For convenience, a Raptean fuzzy number (RFN) is represented by $\tilde{R} = \langle \alpha, \beta, \gamma \rangle$.

Theorem 2.1. *The space of Raptean fuzzy membership grades (RFMGs) is greater than the space of picture fuzzy membership grades (PiFMGs), and spherical fuzzy membership grades (SFMGs).*

Proof: Any point (x, y, z) that is an PiFMG and also a SFMG. For any three numbers, x, y, $z \in [0,1]$, we get $x^3 \leq x^2 \leq x$ and $y^3 \leq y^2 \leq y$. Thus, $x + y + z \leq 1 \Rightarrow x^2 + y^2 + z^2 \leq 1 \Rightarrow x^3 + y^3 + z^3 \leq 1$.

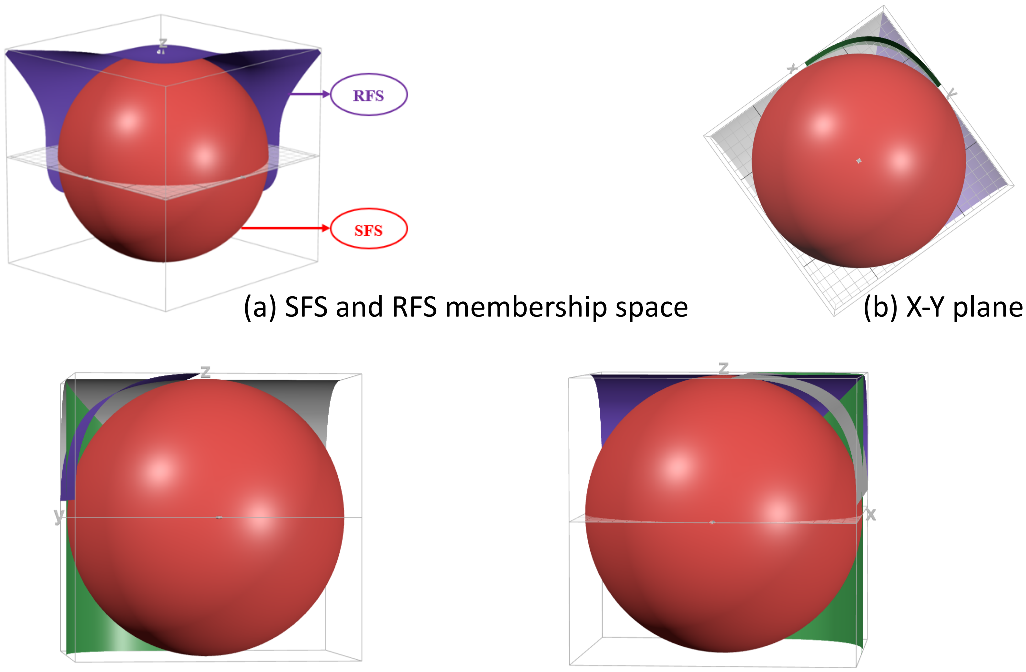
Consider a point, $(\frac{\sqrt[3]{14}}{3}, \frac{\sqrt[3]{5}}{3}, \frac{2}{3})$ where, $\frac{\sqrt[3]{14}}{3}, \frac{\sqrt[3]{5}}{3}, \frac{2}{3} \in [0,1]$. Here, $\left(\frac{\sqrt[3]{14}}{3}\right)^3 + \left(\frac{\sqrt[3]{5}}{3}\right)^3 + \left(\frac{2}{3}\right)^3 = 1$ and therefore, the point lies in the space of RFMGs as the point satisfied the condition of RFSs. But this can be observed that, $\frac{\sqrt[3]{14}}{3} + \frac{\sqrt[3]{5}}{3} + \frac{2}{3} = 2.04 \nleq 1$ and $\left(\frac{\sqrt[3]{14}}{3}\right)^2 + \left(\frac{\sqrt[3]{5}}{3}\right)^2 + \left(\frac{2}{3}\right)^2 = 1.415 \nleq 1$ are not satisfied the conditions of PiFS and SFS, and therefore, the point does not lie in the space of PiFMGs and SFMGs.

This trend becomes apparent when examining Fig. 1 and 2. In the figure, PiFMGs are represented by points lying below the plane $x + y + z \leq 1$, SFMGs by points within the sphere defined by $x^2 + y^2 + z^2 \leq 1$, and RFMGs by points within the region defined by $x^3 + y^3 + z^3 \leq 1$. It's evident that RFMGs accommodate a wider range of membership grades compared to PiFMGs and SFMGs.



(a) IFS, PFS and FFS (b) IFS, PFS, PiFS, and SFS

Fig. 1. Different membership spaces of IFS, PFS, FFS, PiFS, and SFS



(a) SFS and RFS membership space (b) X-Y plane

(c) Y-Z plane (d) Z-X plane

Fig. 2. Geometric Configuration of 3-D membership space of SFS and RFS

3. Algebraic Operations on Raptean Fuzzy Sets

This section presents the basic algebraic operations on Raptean fuzzy sets. Additionally, this study suggests score and accuracy functions for RFSs together with the guidelines for the comparison procedure and examines the theorem with proof.

Definition 3.1. *Let* $\tilde{R} = \{(x, \langle \alpha(x), \beta(x), \gamma(x)\rangle) | \forall x \in U\}$, $\tilde{R}_1 = \{(y, \langle \alpha_1(y), \beta_1(y), \gamma_1(y)\rangle) | \forall y \in U_1\}$ *and* $\tilde{R}_2 = \{(z, \langle \alpha_2(z), \beta_2(z), \gamma_2(z)\rangle) | \forall z \in U_2\}$ *are three RFSs and* $\theta > 0$ *be a scalar. Therefore, the proposed algebraic operations on* $\tilde{R}$, $\tilde{R}_1$ *and* $\tilde{R}_2$ *are as follows:*

(a) $\tilde{R}_1 \oplus \tilde{R}_2 = \langle \sqrt[3]{\alpha_1^3 + \alpha_2^3 - \alpha_1^3\alpha_2^3}, \beta_1\beta_2, \sqrt[3]{(1-\alpha_2^3)\gamma_1^3 + (1-\alpha_1^3)\gamma_2^3 - \gamma_1^3\gamma_2^3} \rangle$

(b) $\tilde{R}_1 \otimes \tilde{R}_2 = \langle \alpha_1\alpha_2, \sqrt[3]{\beta_1^3 + \beta_2^3 - \beta_1^3\beta_2^3}, \sqrt[3]{(1-\beta_2^3)\gamma_1^3 + (1-\beta_1^3)\gamma_2^3 - \gamma_1^3\gamma_2^3} \rangle$

(c) $\tilde{R}^c = \langle \beta(x), \alpha(x), \gamma(x) \rangle$

(d) $\theta \cdot \tilde{R} = \langle \sqrt[3]{1-(1-\alpha^3)^\theta}, \beta^\theta, \sqrt[3]{(1-\alpha^3)^\theta - (1-\alpha^3-\gamma^3)^\theta} \rangle$

(e) $\tilde{R}^\theta = \langle \alpha^\theta, \sqrt[3]{1-(1-\beta^3)^\theta}, \sqrt[3]{(1-\beta^3)^\theta - (1-\beta^3-\gamma^3)^\theta} \rangle$

(f) $\theta^{\tilde{R}} = \langle \theta^{\sqrt[3]{1-\alpha^3}}, \sqrt[3]{1-\theta^{3\beta}}, \sqrt[3]{1-\theta^{3\gamma}} \rangle$

Theorem 3.1. *Let* $\tilde{R} = \{(x, \langle \alpha(x), \beta(x), \gamma(x)\rangle) | \forall x \in U\}$, $\tilde{R}_1 = \{(y, \langle \alpha_1(y), \beta_1(y), \gamma_1(y)\rangle) | \forall y \in U_1\}$

and $\tilde{R}_2 = \{(z, \langle \alpha_2(z), \beta_2(z), \gamma_2(z) \rangle) | \forall z \in U_2\}$ are three RFSs and $\theta, \theta_1, \theta_2 > 0$ are any three scalars. Then the proposed algebraic properties on $\tilde{R}$, $\tilde{R}_1$ and $\tilde{R}_2$ are satisfied as follows:

(a) $\tilde{R}_1 \oplus \tilde{R}_2 = \tilde{R}_2 \oplus \tilde{R}_1$
(b) $\tilde{R}_1 \otimes \tilde{R}_2 = \tilde{R}_2 \otimes \tilde{R}_1$
(c) $\theta \cdot (\tilde{R}_1 \oplus \tilde{R}_2) = \theta \cdot \tilde{R}_1 \oplus \theta \cdot \tilde{R}_2$
(d) $(\theta_1 + \theta_2) \cdot \tilde{R} = \theta_1 \cdot \tilde{R} \oplus \theta_2 \cdot \tilde{R}$
(e) $(\tilde{R}_1 \otimes \tilde{R}_2)^\theta = \tilde{R}_1^\theta \otimes \tilde{R}_2^\theta$
(f) $\tilde{R}^{(\theta_1+\theta_2)} = \tilde{R}^{\theta_1} \otimes \tilde{R}^{\theta_2}$

Proof. For the three RFSs $\tilde{R}$, $\tilde{R}_1$ and $\tilde{R}_2$, and three any real numbers $\theta, \theta_1, \theta_2 > 0$, the proof of the algebraic properties according to *Definition 3.1* can be obtain as follows:

(a) $\tilde{R}_1 \oplus \tilde{R}_2 = \langle \sqrt[3]{\alpha_1^3 + \alpha_2^3 - \alpha_1^3\alpha_2^3}, \beta_1\beta_2, \sqrt[3]{(1-\alpha_2^3)\gamma_1^3 + (1-\alpha_1^3)\gamma_2^3 - \gamma_1^3\gamma_2^3} \rangle$
$= \langle \sqrt[3]{\alpha_2^3 + \alpha_1^3 - \alpha_2^3\alpha_1^3}, \beta_2\beta_1, \sqrt[3]{(1-\alpha_1^3)\gamma_2^3 + (1-\alpha_2^3)\gamma_1^3 - \gamma_2^3\gamma_1^3} \rangle$
$= \tilde{R}_2 \oplus \tilde{R}_1$

(b) $\tilde{R}_1 \otimes \tilde{R}_2 = \langle \alpha_1\alpha_2, \sqrt[3]{\beta_1^3 + \beta_2^3 - \beta_1^3\beta_2^3}, \sqrt[3]{(1-\beta_2^3)\gamma_1^3 + (1-\beta_1^3)\gamma_2^3 - \gamma_1^3\gamma_2^3} \rangle$
$= \langle \alpha_2\alpha_1, \sqrt[3]{\beta_2^3 + \beta_1^3 - \beta_2^3\beta_1^3}, \sqrt[3]{(1-\beta_1^3)\gamma_2^3 + (1-\beta_2^3)\gamma_1^3 - \gamma_2^3\gamma_1^3} \rangle$
$= \tilde{R}_2 \otimes \tilde{R}_1$

(c) $\theta \cdot (\tilde{R}_1 \oplus \tilde{R}_2) = \theta \cdot \langle \sqrt[3]{\alpha_1^3 + \alpha_2^3 - \alpha_1^3\alpha_2^3}, \beta_1\beta_2, \sqrt[3]{(1-\alpha_2^3)\gamma_1^3 + (1-\alpha_1^3)\gamma_2^3 - \gamma_1^3\gamma_2^3} \rangle$
$= \langle \sqrt[3]{1-(1-\alpha_1^3)^\theta(1-\alpha_2^3)^\theta}, \beta_1^\theta\beta_2^\theta, \sqrt[3]{(1-\alpha_1^3)^\theta(1-\alpha_2^3)^\theta - (1-\alpha_1^3-\gamma_1^3)^\theta(1-\alpha_2^3-\gamma_2^3)^\theta} \rangle$
Now,
$\theta \cdot \tilde{R}_1 \oplus \theta \cdot \tilde{R}_2 = \langle \sqrt[3]{1-(1-\alpha_1^3)^\theta}, \beta_1^\theta, \sqrt[3]{(1-\alpha_1^3)^\theta - (1-\alpha_1^3-\gamma_1^3)^\theta} \rangle \oplus$
$\langle \sqrt[3]{1-(1-\alpha_2^3)^\theta}, \beta_2^\theta, \sqrt[3]{(1-\alpha_2^3)^\theta - (1-\alpha_2^3-\gamma_2^3)^\theta} \rangle$
$= \langle \sqrt[3]{1-(1-\alpha_1^3)^\theta(1-\alpha_2^3)^\theta}, \beta_1^\theta\beta_2^\theta, \sqrt[3]{(1-\alpha_1^3)^\theta(1-\alpha_2^3)^\theta - (1-\alpha_1^3-\gamma_1^3)^\theta(1-\alpha_2^3-\gamma_2^3)^\theta} \rangle$
Therefore, $\theta \cdot (\tilde{R}_1 \oplus \cdot\tilde{R}_2) = \theta \cdot \tilde{R}_1 \oplus \theta \cdot \tilde{R}_2$

(d) $(\theta_1 + \theta_2) \cdot \tilde{R} = \langle \sqrt[3]{1-(1-\alpha^3)^{(\theta_1+\theta_2)}}, \beta^{(\theta_1+\theta_2)}, \sqrt[3]{(1-\alpha^3)^{(\theta_1+\theta_2)} - (1-\alpha^3-\gamma^3)^{(\theta_1+\theta_2)}} \rangle$
Now,
$\theta_1 \cdot \tilde{R} \oplus \theta_2 \cdot \tilde{R} = \langle \sqrt[3]{1-(1-\alpha^3)^{\theta_1}}, \beta^{\theta_1}, \sqrt[3]{(1-\alpha^3)^{\theta_1} - (1-\alpha^3-\gamma^3)^{\theta_1}} \rangle \oplus$
$\langle \sqrt[3]{1-(1-\alpha^3)^{\theta_2}}, \beta^{\theta_2}, \sqrt[3]{(1-\alpha^3)^{\theta_2} - (1-\alpha^3-\gamma^3)^{\theta_2}} \rangle$
$= \langle \sqrt[3]{1-(1-\alpha^3)^{(\theta_1+\theta_2)}}, \beta^{(\theta_1+\theta_2)}, \sqrt[3]{(1-\alpha^3)^{(\theta_1+\theta_2)} - (1-\alpha^3-\gamma^3)^{(\theta_1+\theta_2)}} \rangle$
Therefore, $(\theta_1 + \theta_2) \cdot \tilde{R} = \theta_1 \cdot \tilde{R} \oplus \theta_2 \cdot \tilde{R}$

(e) $(\tilde{R}_1 \otimes \tilde{R}_2)^\theta = (\langle \alpha_1\alpha_2, \sqrt[3]{\beta_1^3 + \beta_2^3 - \beta_1^3\beta_2^3}, \sqrt[3]{(1-\beta_2^3)\gamma_1^3 + (1-\beta_1^3)\gamma_2^3 - \gamma_1^3\gamma_2^3} \rangle)^\theta$
$= \langle \alpha_1^\theta\alpha_2^\theta, \sqrt[3]{1-(1-\beta_1^3)^\theta(1-\beta_2^3)^\theta}, \sqrt[3]{(1-\beta_1^3)^\theta(1-\beta_2^3)^\theta - (1-\beta_1^3-\gamma_1^3)^\theta(1-\beta_2^3-\gamma_2^3)^\theta} \rangle$
Now,
$\tilde{R}_1^\theta \otimes \tilde{R}_2^\theta = \langle \alpha_1^\theta, \sqrt[3]{1-(1-\beta_1^3)^\theta}, \sqrt[3]{(1-\beta_1^3)^\theta - (1-\beta_1^3-\gamma_1^3)^\theta} \rangle \otimes$
$\langle \alpha_2^\theta, \sqrt[3]{1-(1-\beta_2^3)^\theta}, \sqrt[3]{(1-\beta_2^3)^\theta - (1-\beta_2^3-\gamma_2^3)^\theta} \rangle$
$= \langle \alpha_1^\theta\alpha_2^\theta, \sqrt[3]{1-(1-\beta_1^3)^\theta(1-\beta_2^3)^\theta}, \sqrt[3]{(1-\beta_1^3)^\theta(1-\beta_2^3)^\theta - (1-\beta_1^3-\gamma_1^3)^\theta(1-\beta_2^3-\gamma_2^3)^\theta} \rangle$
Therefore, $(\tilde{R}_1 \otimes \tilde{R}_2)^\theta = \tilde{R}_1^\theta \otimes \tilde{R}_2^\theta$

(f) $\tilde{R}^{(\theta_1+\theta_2)} = \langle \alpha^{(\theta_1+\theta_2)}, \sqrt[3]{1-(1-\beta^3)^{(\theta_1+\theta_2)}}, \sqrt[3]{(1-\beta^3)^{(\theta_1+\theta_2)} - (1-\beta^3-\gamma^3)^{(\theta_1+\theta_2)}} \rangle$
Now,
$\tilde{R}^{\theta_1} \otimes \tilde{R}^{\theta_2} = \langle \alpha^{\theta_1}, \sqrt[3]{1-(1-\beta^3)^{\theta_1}}, \sqrt[3]{(1-\beta^3)^{\theta_1} - (1-\beta^3-\gamma^3)^{\theta_1}} \rangle \otimes$
$\langle \alpha^{\theta_2}, \sqrt[3]{1-(1-\beta^3)^{\theta_2}}, \sqrt[3]{(1-\beta^3)^{\theta_2} - (1-\beta^3-\gamma^3)^{\theta_2}} \rangle$
$= \langle \alpha^{(\theta_1+\theta_2)}, \sqrt[3]{1-(1-\beta^3)^{(\theta_1+\theta_2)}}, \sqrt[3]{(1-\beta^3)^{(\theta_1+\theta_2)} - (1-\beta^3-\gamma^3)^{(\theta_1+\theta_2)}} \rangle$
Therefore, $\tilde{R}^{(\theta_1+\theta_2)} = \tilde{R}^{\theta_1} \otimes \tilde{R}^{\theta_2}$

□

Definition 3.2. *Let $\tilde{R} = \langle \alpha, \beta, \gamma \rangle$ be a RFN. Then, in order to defuzzify the RFNs for the objective of ranking, the proposed score function of $\tilde{R}$ is stated as,*

$$Score(\tilde{R}) = \mathcal{S}(\tilde{R}) = (\alpha^3 - \beta^3)(1 - \gamma^3) \in [-1, 1] \quad (3)$$

Definition 3.3. *Let $\tilde{R} = \langle \alpha, \beta, \gamma \rangle$ be a RFN. Then, in order to defuzzify the RFNs for the objective of ranking, the proposed accuracy function of $\tilde{R}$ can be stated as,*

$$Accuracy(\tilde{R}) = \mathcal{A}(\tilde{R}) = \alpha^3 + \beta^3 + \gamma^3 \in [0, 1] \tag{4}$$

clearly, $\mathcal{A}(\tilde{R}) \in [0, 1]$ as $0 \leq \alpha^3 + \beta^3 + \gamma^3 \leq 1$, from the Definition 2.1.

Theorem 3.2. *For any RFN $\tilde{R} = \langle \alpha, \beta, \gamma \rangle$, the proposed score function, Score$(\tilde{R}) \in [-1, 1]$.*

Proof. From the *Definition 2.1* it is clear that for any RFN $\tilde{R} = \langle \alpha, \beta, \gamma \rangle$, the MGs $\alpha, \beta, \gamma \in [0, 1]$ and meets the constraint $0 \leq \alpha^3 + \beta^3 + \gamma^3 \leq 1$. Then we have,

$$\begin{aligned}
\mathcal{S}(\tilde{R}) &= (\alpha^3 - \beta^3)(1 - \gamma^3) \\
&< (\alpha^3 + \beta^3 + \gamma^3)(1 - \gamma^3), \text{ as } \alpha, \beta, \gamma > 0 \\
&\leq 1 \cdot (1 - \gamma^3), \text{ as } \alpha^3 + \beta^3 + \gamma^3 \leq 1 \\
&\leq 1, \text{ as } \gamma \in [0, 1]
\end{aligned}$$

Again,

$$\begin{aligned}
\mathcal{S}(\tilde{R}) &= (\alpha^3 - \beta^3)(1 - \gamma^3) \\
&> (-\alpha^3 - \beta^3 - \gamma^3)(1 - \gamma^3), \text{ as } \alpha, \beta, \gamma > 0 \\
&> -(\alpha^3 + \beta^3 + \gamma^3)(1 - \gamma^3) \\
&\geq -1 \cdot (1 - \gamma^3), \text{ as } \alpha^3 + \beta^3 + \gamma^3 \leq 1 \\
&\geq -1, \text{ as } \gamma \in [0, 1]
\end{aligned}$$

Hence, $-1 \leq \mathcal{S}(\tilde{R}) \leq 1$.

Therefore, for any RFN $\tilde{R} = \langle \alpha, \beta, \gamma \rangle$, the score function, *Score*$(\tilde{R}) \in [-1, 1]$ and it is obvious that a greater *Score*$(\tilde{R})$ corresponds to a larger $\tilde{R}$. Specifically, when $\tilde{R} = \langle 1, 0, 0 \rangle$, then $\mathcal{S}(\tilde{R}) = 1$ has the highest value, and when $\tilde{R} = \langle 0, 1, 0 \rangle$, then $\mathcal{S}(\tilde{R}) = -1$ has the smallest value. □

Theorem 3.3. *The score function $\mathcal{S}(\tilde{R})$, defined in Definition 3.2 is monotonically increasing with respect to α and monotonically decreasing with respect to β and oscillates with respect to γ.*

Proof. Applying partial differentiation of both sides of Eq. (3) with respect to α, gives

$$\frac{\partial \mathcal{S}(\tilde{R})}{\partial \alpha} = 3\alpha^2(1 - \gamma^3) \geq 0, \forall \alpha, \beta, \gamma \in [0, 1]$$

Similarly, applying partial differentiation of both sides of Eq. (3) with respect to β and γ, gives

$$\frac{\partial \mathcal{S}(\tilde{R})}{\partial \beta} = -3\beta^2(1 - \gamma^3) \leq 0, \forall \alpha, \beta, \gamma \in [0, 1]$$

$$\begin{aligned}
\frac{\partial \mathcal{S}(\tilde{R})}{\partial \gamma} = -3\gamma^2(\alpha^3 - \beta^3) &\leq 0, \text{ when } \alpha \geq \beta \\
&\geq 0, \text{ when } \alpha \leq \beta
\end{aligned}$$

This concludes the proof of the theorem. □

Theorem 3.4. *For any two RFNs $\tilde{R}_1 = \langle \alpha_1, \beta_1, \gamma_1 \rangle$ and $\tilde{R}_2 = \langle \alpha_2, \beta_2, \gamma_2 \rangle$, if $\alpha_1 > \alpha_2$, $\beta_1 < \beta_2$ and $\gamma_1 < \gamma_2$ then must be $\mathcal{S}(\tilde{R}_1) > \mathcal{S}(\tilde{R}_2)$.*

Proof. Let $\tilde{R}_1 = \langle \alpha_1, \beta_1, \gamma_1 \rangle$ and $\tilde{R}_2 = \langle \alpha_2, \beta_2, \gamma_2 \rangle$ are two RFNs. Such that, $\alpha_1 > \alpha_2$, $\beta_1 < \beta_2$ and $\gamma_1 < \gamma_2$.

Since, $\alpha_1, \alpha_2, \beta_1, \beta_2 \in [0, 1]$, then $\alpha_1^3 > \alpha_2^3, \beta_1^3 < \beta_2^3$
$\Rightarrow \alpha_1^3 > \alpha_2^3, -\beta_1^3 > -\beta_2^3$
Adding we have, $\alpha_1^3 - \beta_1^3 > \alpha_2^3 - \beta_2^3$
Again since, $\gamma_1, \gamma_2 \in [0, 1]$, then $\gamma_1^3 < \gamma_2^3$
$\Rightarrow -\gamma_1^3 > -\gamma_2^3 \Rightarrow 1 - \gamma_1^3 > 1 - \gamma_2^3$
Multiplying both cases, we have,
$(\alpha_1^3 - \beta_1^3)(1 - \gamma_1^3) > (\alpha_2 - \beta_2^3)(1 - \gamma_2^3)$
$\Rightarrow \mathcal{S}(\tilde{R}_1) > \mathcal{S}(\tilde{R}_2)$
This concludes the proof of the theorem. □

Additionally, the following *Example 3.1* has been used to validate the applicability of the theorem.

Example 3.1. *Let there are two RFNs $\tilde{R}_1 = \langle 0.9, 0.2, 0.3 \rangle$ and $\tilde{R}_2 = \langle 0.7, 0.3, 0.4 \rangle$. From Definition 3.2, $\mathcal{S}(\tilde{R}_1) = 0.702$ and $\mathcal{S}(\tilde{R}_2) = 0.296$. That is, $\mathcal{S}(\tilde{R}_1) > \mathcal{S}(\tilde{R}_2)$. Therefore, it could be stated that if $\alpha_1 > \alpha_2$, $\beta_1 < \beta_2$ and $\gamma_1 < \gamma_2$ then must be $\mathcal{S}(\tilde{R}_1) > \mathcal{S}(\tilde{R}_2)$.*

Theorem 3.5. *For any two RFNs $\tilde{R}_1 = \langle \alpha_1, \beta_1, \gamma_1 \rangle$ and $\tilde{R}_2 = \langle \alpha_2, \beta_2, \gamma_2 \rangle$, if $\mathcal{S}(\tilde{R}_1) > \mathcal{S}(\tilde{R}_2)$ then it is not obvious that $\alpha_1 > \alpha_2$, $\beta_1 < \beta_2$ and $\gamma_1 < \gamma_2$.*

The following *Examples 3.2*, *3.3* and *3.4* have been used to validate the applicability of the theorem.

Example 3.2. *Let there are two RFNs $\tilde{R}_1 = \langle 0.8, 0.2, 0.5 \rangle$ and $\tilde{R}_2 = \langle 0.7, 0.5, 0.4 \rangle$. From Definition 3.2, $\mathcal{S}(\tilde{R}_1) = 0.441$ and $\mathcal{S}(\tilde{R}_2) = 0.204$. That is, $\mathcal{S}(\tilde{R}_1) > \mathcal{S}(\tilde{R}_2)$. Here, $\alpha_1 > \alpha_2$, $\beta_1 < \beta_2$ but $\gamma_1 > \gamma_2$. Therefore, it could be stated that if $\mathcal{S}(\tilde{R}_1) > \mathcal{S}(\tilde{R}_2)$ then it is not obvious that $\alpha_1 > \alpha_2$, $\beta_1 < \beta_2$ and $\gamma_1 < \gamma_2$.*

Example 3.3. *Let there are two RFNs $\tilde{R}_1 = \langle 0.75, 0.25, 0.1 \rangle$ and $\tilde{R}_2 = \langle 0.8, 0.2, 0.7 \rangle$. From Definition 3.2, $\mathcal{S}(\tilde{R}_1) = 0.406$ and $\mathcal{S}(\tilde{R}_2) = 0.331$. Here, $\mathcal{S}(\tilde{R}_1) > \mathcal{S}(\tilde{R}_2)$ but $\alpha_1 < \alpha_2$, $\beta_1 > \beta_2$ but $\gamma_1 < \gamma_2$. Therefore, it could be stated that if $\mathcal{S}(\tilde{R}_1) > \mathcal{S}(\tilde{R}_2)$ then it is not obvious that $\alpha_1 > \alpha_2$, $\beta_1 < \beta_2$ and $\gamma_1 < \gamma_2$.*

Example 3.4. *Let there are two RFNs $\tilde{R}_1 = \langle 0.6, 0.4, 0.1 \rangle$ and $\tilde{R}_2 = \langle 0.6, 0.4, 0.5 \rangle$. From Definition 3.2, $\mathcal{S}(\tilde{R}_1) = 0.152$ and $\mathcal{S}(\tilde{R}_2) = 0.133$. Here, $\mathcal{S}(\tilde{R}_1) > \mathcal{S}(\tilde{R}_2)$ but $\alpha_1 = \alpha_2$, $\beta_1 = \beta_2$ and $\gamma_1 < \gamma_2$. Therefore, it could be stated that if $\mathcal{S}(\tilde{R}_1) > \mathcal{S}(\tilde{R}_2)$ then it is not obvious that $\alpha_1 > \alpha_2$, $\beta_1 < \beta_2$ and $\gamma_1 < \gamma_2$.*

In the following, *Definition 3.4* and *3.5* define the comparison procedure of Raptean fuzzy sets for the objective of ranking.

Definition 3.4. *Let $\tilde{R}_i = \langle \alpha_i, \beta_i, \gamma_i \rangle$, $i = 1, 2, \ldots n$ be a group of "n" RFNs. Then $\tilde{R}_{min}$ and $\tilde{R}_{max}$ from the collection of $\tilde{R}_i, \forall i = 1, 2, \ldots, n$ is stated as,*

(a) $\tilde{R}_{min} = \langle min\{\alpha_i\}, max\{\beta_i\}, max\{\gamma_i\} \rangle, \forall i$
(b) $\tilde{R}_{max} = \langle max\{\alpha_i\}, min\{\beta_i\}, min\{\gamma_i\} \rangle, \forall i$

Definition 3.5. *For any two RFNs $\tilde{R}_1 = \langle \alpha_1, \beta_1, \gamma_1 \rangle$ and $\tilde{R}_2 = \langle \alpha_2, \beta_2, \gamma_2 \rangle$, the comparative procedures are as follows:*

(a) If $\mathcal{S}(\tilde{R}_1) < \mathcal{S}(\tilde{R}_2)$ then $\tilde{R}_1 < \tilde{R}_2$

(b) *If* $\mathcal{S}(\tilde{R}_1) > \mathcal{S}(\tilde{R}_2)$ *then* $\tilde{R}_1 > \tilde{R}_2$
(c) *If* $\mathcal{S}(\tilde{R}_1) = \mathcal{S}(\tilde{R}_2)$ *then,*
(i) *If* $\mathcal{A}(\tilde{R}_1) < \mathcal{A}(\tilde{R}_2)$ *then* $\tilde{R}_1 < \tilde{R}_2$
(ii) *If* $\mathcal{A}(\tilde{R}_1) > \mathcal{A}(\tilde{R}_2)$ *then* $\tilde{R}_1 > \tilde{R}_2$
(iii) *If* $\mathcal{A}(\tilde{R}_1) = \mathcal{A}(\tilde{R}_2)$ *then* $\tilde{R}_1 = \tilde{R}_2$

4. Aggregation of Raptean Fuzzy Sets

Definition 4.1. *Let* $\tilde{R}_i = \langle \alpha_i, \beta_i, \gamma_i \rangle, i = 1, 2, \ldots n$ *be a group of "*n*" RFNs. Then the proposed Raptean fuzzy weighted arithmetic mean (RFWAM) operator is a function:* $\tilde{R}^n \to \tilde{R}$*, and is stated as,*

$$RFWAM(\tilde{R}_1, \ldots, \tilde{R}_n) = \oplus_{i=1}^{n} \theta_i \tilde{R}_i = \langle \sqrt[3]{1 - \prod_{i=1}^{n}(1-\alpha_i^3)^{\theta_i}},\ \prod_{i=1}^{n} \beta_i^{\theta_i},\ \sqrt[3]{\prod_{i=1}^{n}(1-\alpha_i^3)^{\theta_i} - \prod_{i=1}^{n}(1-\alpha_i^3-\gamma_i^3)^{\theta_i}} \rangle \tag{5}$$

where, $\theta_i (i = 1, 2, \ldots, n)$ *are the weights of* $\tilde{R}_i (i = 1, 2, \ldots, n)$ *with* $\theta_i \in [0, 1]$ *and* $\sum_{i=1}^{n} \theta_i = 1.$

Proof. To proof of the *Definition 4.1* we apply the mathematical induction on "n", so we can proceed as follows:

We are showing the proof separately for the MG, NMG, and HG of RFNs.

For $n = 2$ we have

$$RFWAM(\tilde{R}_1, \tilde{R}_2) = \theta_1 \tilde{R}_1 \oplus \theta_2 \tilde{R}_2$$
$$= \langle \sqrt[3]{1-(1-\alpha_1^3)^{\theta_1}}, \beta_1^{\theta_1}, \sqrt[3]{(1-\alpha_1^3)^{\theta_1} - (1-\alpha_1^3-\gamma_1^3)^{\theta_1}} \rangle \oplus$$
$$\langle \sqrt[3]{1-(1-\alpha_2^3)^{\theta_2}}, \beta_2^{\theta_2}, \sqrt[3]{(1-\alpha_2^3)^{\theta_2} - (1-\alpha_2^3-\gamma_2^3)^{\theta_2}} \rangle$$
$$= \langle \sqrt[3]{1-(1-\alpha_1^3)^{\theta_1}(1-\alpha_2^3)^{\theta_2}}, \beta_1^{\theta_1}\beta_2^{\theta_2}, \sqrt[3]{(1-\alpha_1^3)^{\theta_1}(1-\alpha_2^3)^{\theta_2} - (1-\alpha_1^3-\gamma_1^3)^{\theta_1}(1-\alpha_2^3-\gamma_2^3)^{\theta_2}} \rangle$$
$$= \langle \sqrt[3]{1-\prod_{i=1}^{2}(1-\alpha_i^3)^{\theta_i}}, \prod_{i=1}^{2}\beta_i^{\theta_i}, \sqrt[3]{\prod_{i=1}^{2}(1-\alpha_i^3)^{\theta_i} - \prod_{i=1}^{2}(1-\alpha_i^3-\gamma_i^3)^{\theta_i}} \rangle$$

Therefore, the *Definition 4.1* is true for $n = 2$.

Now, assume that the *Definition 4.1* is true for $n = k$.

Therefore, we have

$$RFWAM(\tilde{R}_1, \ldots, \tilde{R}_k) = \oplus_{i=1}^{k} \theta_i \tilde{R}_i$$
$$= \langle \sqrt[3]{1-\prod_{i=1}^{k}(1-\alpha_i^3)^{\theta_i}}, \prod_{i=1}^{k}\beta_i^{\theta_i}, \sqrt[3]{\prod_{i=1}^{k}(1-\alpha_i^3)^{\theta_i} - \prod_{i=1}^{k}(1-\alpha_i^3-\gamma_i^3)^{\theta_i}} \rangle$$

Now, we have to prove that the *Definition 4.1* is true for $n = k + 1$.

$$RFWAM(\tilde{R}_1, \ldots, \tilde{R}_k, \tilde{R}_{k+1}) = \oplus_{i=1}^{k} \theta_i \tilde{R}_i \oplus \theta_{k+1}\tilde{R}_{k+1}$$
$$= \langle \sqrt[3]{1-\prod_{i=1}^{k}(1-\alpha_i^3)^{\theta_i}}, \prod_{i=1}^{k}\beta_i^{\theta_i}, \sqrt[3]{\prod_{i=1}^{k}(1-\alpha_i^3)^{\theta_i} - \prod_{i=1}^{k}(1-\alpha_i^3-\gamma_i^3)^{\theta_i}} \rangle \oplus$$
$$\langle \sqrt[3]{1-(1-\alpha_{k+1}^3)^{\theta_{k+1}}}, \beta_{k+1}^{\theta_{k+1}}, \sqrt[3]{(1-\alpha_{k+1}^3)^{\theta_{k+1}} - (1-\alpha_{k+1}^3-\gamma_{k+1}^3)^{\theta_{k+1}}} \rangle$$
$$= \langle \sqrt[3]{1-\prod_{i=1}^{k}(1-\alpha_i^3)^{\theta_i}(1-\alpha_{k+1}^3)^{\theta_{k+1}}}, \prod_{i=1}^{k}\beta_i^{\theta_i}\beta_{k+1}^{\theta_{k+1}},$$
$$\sqrt[3]{\prod_{i=1}^{k}(1-\alpha_i^3)^{\theta_i}(1-\alpha_{k+1}^3)^{\theta_{k+1}} - \prod_{i=1}^{k}(1-\alpha_i^3-\gamma_i^3)^{\theta_1}(1-\alpha_{k+1}^3-\gamma_{k+1}^3)^{\theta_{k+1}}} \rangle$$
$$= \langle \sqrt[3]{1-\prod_{i=1}^{k+1}(1-\alpha_i^3)^{\theta_i}}, \prod_{i=1}^{k+1}\beta_i^{\theta_i}, \sqrt[3]{\prod_{i=1}^{k+1}(1-\alpha_i^3)^{\theta_i} - \prod_{i=1}^{k+1}(1-\alpha_i^3-\gamma_i^3)^{\theta_i}} \rangle$$

Thus, the *Definition 4.1* is valid for $n = k + 1$, we can use mathematical induction to conclude that the definition of the RFWAM operator is valid for all n. □

Property 4.1. *Here, we've listed three characteristics that the RFWAM operator satisfies:*

(a) *Idempotency: If* $\tilde{R}_i = \{(x, \langle \alpha_i(x), \beta_i(x), \gamma_i(x) \rangle) | \forall x \in U_i\}$ $(i = 1, 2, ..., n)$ *be any grouping of RFSs and if* $\tilde{R}_i = \tilde{R}, \forall i$ *then* $RFWAM(\tilde{R}_1, \tilde{R}_2, \ldots, \tilde{R}_n) = \tilde{R}.$

(b) *Boundedness: If* $\tilde{R}_i = \{(x, \langle (\alpha_i(x), \beta_i(x), \gamma_i(x) \rangle) | \forall x \in U_i\}$ $(i = 1, 2, ..., n)$ *be any collection of RFSs and if* $\tilde{R}_{min} = min\{\tilde{R}_1, \tilde{R}_2, \ldots, \tilde{R}_n\}$ *and* $\tilde{R}_{max} = max\{\tilde{R}_1, \tilde{R}_2, \ldots, \tilde{R}_n\}$ *and if* $\tilde{R}^- =$

$RFWAM(min\{\tilde{R}_1\}, min\{\tilde{R}_2\}, \ldots, min\{\tilde{R}_n\})$ *and* $\tilde{R}^+ = RFWAM(max\{\tilde{R}_1\}, max\{\tilde{R}_2\}, \ldots, max\{\tilde{R}_n\})$*, then* $\tilde{R}^- \leq RFWAM(\tilde{R}_1, \tilde{R}_2, \ldots, \tilde{R}_n) \leq \tilde{R}^+$.

(c) *Monotonicity: If* $\tilde{R}_i = \{(x, \langle \alpha_i(x), \beta_i(x), \gamma_i(x) \rangle) | \forall x \in U_i\}$ $(i = 1, 2, ..., n)$ *and* $\tilde{R}_i^* = \{(x, \langle \alpha_i^*(x), \beta_i^*(x), \gamma_i^*(x) \rangle) | \forall x \in U_i\}$ $(i = 1, 2, ..., n)$ *be any grouping of RFSs and if* $\tilde{R}_i \leq \tilde{R}_i^*$ $\forall i$*, then* $RFWAM(\tilde{R}_1, \ldots, \tilde{R}_n) \leq RFWAM(\tilde{R}_1^*, \ldots, \tilde{R}_n^*)$.

Proof. (a) Since $\tilde{R}_i = \tilde{R} = \langle \alpha, \beta, \gamma \rangle \forall i$ and $\sum_{i=1}^n \theta_i = 1$, then, we have the following from the *Definition 4.1*:

$$\begin{aligned}
&RFWAM(\tilde{R}_1, \ldots, \tilde{R}_n) = \oplus_{i=1}^n \theta_i \tilde{R}_i \\
&= \langle \sqrt[3]{1 - \prod_{i=1}^n (1 - \alpha_i^3)^{\theta_i}},\ \prod_{i=1}^n \beta_i^{\theta_i},\ \sqrt[3]{\prod_{i=1}^n (1 - \alpha_i^3)^{\theta_i} - \prod_{i=1}^n (1 - \alpha_i^3 - \gamma_i^3)^{\theta_i}} \rangle \\
&= \langle \sqrt[3]{1 - (1 - \alpha^3)^{\sum_{i=1}^n \theta_i}},\ \beta^{\sum_{i=1}^n \theta_i},\ \sqrt[3]{(1 - \alpha^3)^{\sum_{i=1}^n \theta_i} - (1 - \alpha^3 - \gamma^3)^{\sum_{i=1}^n \theta_i}} \rangle \\
&= \langle \sqrt[3]{1 - (1 - \alpha^3)},\ \beta,\ \sqrt[3]{(1 - \alpha^3) - (1 - \alpha^3 - \gamma^3)} \rangle \\
&= \langle \sqrt[3]{1 - 1 + \alpha^3)},\ \beta,\ \sqrt[3]{1 - \alpha^3 - 1 + \alpha^3 + \gamma^3)} \rangle \\
&= \langle \sqrt[3]{\alpha^3)},\ \beta,\ \sqrt[3]{\gamma^3)} \rangle = \langle \alpha, \beta, \gamma \rangle = \tilde{R}
\end{aligned}$$

(b) From *Definition 3.5*, $\tilde{R}_{min} = min\{\tilde{R}_1, \tilde{R}_2, \ldots, \tilde{R}_n\} = min\{\tilde{R}_i\} = \langle min\{\alpha_i\}, max\{\beta_i\}, max\{\gamma_i\} \rangle, \forall i$ and $\tilde{R}_{max} = max\{\tilde{R}_1, \tilde{R}_2, \ldots, \tilde{R}_n\} = max\{\tilde{R}_i\} = \langle max\{\alpha_i\}, min\{\beta_i\}, min\{\gamma_i\} \rangle, \forall i$.

Then, from the idempotency property, we have:

$$\begin{aligned}
\tilde{R}^- &= RFWAM(min\{\tilde{R}_1\}, min\{\tilde{R}_2\}, \ldots, min\{\tilde{R}_n\}) \\
&= RFWAM(min\{\tilde{R}_i\}), \forall i \\
\text{and } \tilde{R}^+ &= RFWAM(max\{\tilde{R}_1\}, max\{\tilde{R}_2\}, \ldots, max\{\tilde{R}_n\}) \\
&= RFWAM(max\{\tilde{R}_i\}), \forall i
\end{aligned}$$

Now since $\tilde{R}_{min} \leq \tilde{R}_i \leq \tilde{R}_{max}, \forall i$ we can write:

$$\tilde{R}^- \leq RFWAM(\tilde{R}_1, \tilde{R}_2, \ldots, \tilde{R}_n) \leq \tilde{R}^+$$

(c) Since, $\tilde{R}_i \leq \tilde{R}_i^*$ $\forall i$, then from idempotency and boundedness property we can write:

$$RFWAM(\tilde{R}_1, \tilde{R}_2, \ldots, \tilde{R}_n) \leq RFWAM(\tilde{R}_1^*, \tilde{R}_2^*, \ldots, \tilde{R}_n^*)$$

□

Definition 4.2. *Let* $\tilde{R}_i = \langle \alpha_i, \beta_i, \gamma_i \rangle$*,* $i = 1, 2, \ldots n$ *be a group of "n" RFNs. Then the proposed Raptean fuzzy weighted geometric mean (RFWGM) operator is a mapping:* $\tilde{R}^n \rightarrow \tilde{R}$*, and is stated as,*

$$\begin{aligned}
RFWGM(\tilde{R}_1, \ldots, \tilde{R}_n) &= \otimes_{i=1}^n \left(\tilde{R}_i\right)^{\theta_i} \\
&= \langle \prod_{i=1}^n \alpha_i^{\theta_i},\ \sqrt[3]{1 - \prod_{i=1}^n (1 - \beta_i^3)^{\theta_i}},\ \sqrt[3]{\prod_{i=1}^n (1 - \beta_i^3)^{\theta_i} - \prod_{i=1}^n (1 - \beta_i^3 - \gamma_i^3)^{\theta_i}} \rangle
\end{aligned} \tag{6}$$

where, $\theta_i (i = 1, 2, \ldots, n)$ *are the weights of* $\tilde{R}_i (i = 1, 2, \ldots, n)$ *with* $\theta_i \in [0, 1]$ *and* $\sum_{i=1}^n \theta_i = 1$.

Proof. The proof is omitted here because it is the same as the proof of *Definition 4.1*. □

Similarly, the RFWGM operator satisfies the idempotency, boundedness, and monotonicity properties, so they are omitted here.

Definition 4.3. *If* $\tilde{R}_1 = \langle \alpha_1, \beta_1, \gamma_1 \rangle$ *and* $\tilde{R}_2 = \langle \alpha_2, \beta_2, \gamma_2 \rangle$ *are any two RFNs. Then, the proposed distance between* $\tilde{R}_1$ *and* $\tilde{R}_2$ *defined as follows:*

(a) Euclidean distance:

$$D^E = \sqrt{\frac{1}{2}\left[(\alpha_1^3 - \alpha_2^3)^2 + (\beta_1^3 - \beta_2^3)^2 + (\gamma_1^3 - \gamma_2^3)^2\right]} \tag{7}$$

(b) Zhang and Xu distance:

$$D^{ZX} = \frac{1}{2}\left(|\alpha_1^3 - \alpha_2^3| + |\beta_1^3 - \beta_2^3| + |\gamma_1^3 - \gamma_2^3|\right) \tag{8}$$

Theorem 4.1. *If $\tilde{R}_1 = \langle \alpha_1, \beta_1, \gamma_1 \rangle$ and $\tilde{R}_2 = \langle \alpha_2, \beta_2, \gamma_2 \rangle$ are any two RFNs. Then,*

(a) Euclidean distance:
- *(i)* $0 \le D^E \le 1$
- *(ii)* $D^E(\tilde{R}_1, \tilde{R}_2) = D^E(\tilde{R}_2, \tilde{R}_1)$
- *(iii)* $D^E(\tilde{R}_1, \tilde{R}_2) = 0$*, if and only if* $\tilde{R}_1 = \tilde{R}_2$

(b) Zhang and Xu distance:
- *(i)* $0 \le D^{ZX} \le 1$
- *(ii)* $D^{ZX}(\tilde{R}_1, \tilde{R}_2) = D^{ZX}(\tilde{R}_2, \tilde{R}_1)$
- *(iii)* $D^{ZX}(\tilde{R}_1, \tilde{R}_2) = 0$*, if and only if* $\tilde{R}_1 = \tilde{R}_2$

Proof. Let $\tilde{R}_1 = \langle \alpha_1, \beta_1, \gamma_1 \rangle$ and $\tilde{R}_2 = \langle \alpha_2, \beta_2, \gamma_2 \rangle$ are any two RFNs. Then, from *Definition 2.1* we have $\alpha_i, \beta_i, \gamma_i \in [0,1]$ and satisfy the condition $0 \le \alpha_i^3 + \beta_i^3 + \gamma_i^3 \le 1, \forall i = 1, 2$.

(a) Euclidean distance:

(i)
$$\begin{aligned}
D^E &= \sqrt{\tfrac{1}{2}\left[(\alpha_1^3 - \alpha_2^3)^2 + (\beta_1^3 - \beta_2^3)^2 + (\gamma_1^3 - \gamma_2^3)^2\right]} \\
&< \sqrt{\tfrac{1}{2}\left[(\alpha_1^3 + \alpha_2^3)^2 + (\beta_1^3 + \beta_2^3)^2 + (\gamma_1^3 + \gamma_2^3)^2\right]} \\
&< \sqrt{\tfrac{1}{2}\left[(\alpha_1^3 + \alpha_2^3) + (\beta_1^3 + \beta_2^3) + (\gamma_1^3 + \gamma_2^3)\right]} \\
&= \sqrt{\tfrac{1}{2}\left[(\alpha_1^3 + \beta_1^3 + \gamma_1^3) + (\alpha_2^3 + \beta_2^3 + \gamma_2^3)\right]} \\
&\le \sqrt{\tfrac{1}{2}(1+1)} \\
&\le 1
\end{aligned}$$

Again,

$D^E = \sqrt{\tfrac{1}{2}\left[(\alpha_1^3 - \alpha_2^3)^2 + (\beta_1^3 - \beta_2^3)^2 + (\gamma_1^3 - \gamma_2^3)^2\right]} \ge 0,$ as $\alpha_i, \beta_i, \gamma_i \in [0,1], \forall i = 1, 2.$

Therefore, $0 \le D^E \le 1$

(ii)
$$D^E(\tilde{R}_1, \tilde{R}_2) = \sqrt{\tfrac{1}{2}\left[(\alpha_1^3 - \alpha_2^3)^2 + (\beta_1^3 - \beta_2^3)^2 + (\gamma_1^3 - \gamma_2^3)^2\right]}$$
$$\begin{aligned}
D^E(\tilde{R}_2, \tilde{R}_1) &= \sqrt{\tfrac{1}{2}\left[(\alpha_2^3 - \alpha_1^3)^2 + (\beta_2^3 - \beta_1^3)^2 + (\gamma_2^3 - \gamma_1^3)^2\right]} \\
&= \sqrt{\tfrac{1}{2}\left[(\alpha_1^3 - \alpha_2^3)^2 + (\beta_1^3 - \beta_2^3)^2 + (\gamma_1^3 - \gamma_2^3)^2\right]}
\end{aligned}$$

Therefore,

$D^E(\tilde{R}_1, \tilde{R}_2) = D^E(\tilde{R}_2, \tilde{R}_1)$

(iii) $D^E(\tilde{R}_1, \tilde{R}_2) = 0$

$\Rightarrow \sqrt{\tfrac{1}{2}\left[(\alpha_1^3 - \alpha_2^3)^2 + (\beta_1^3 - \beta_2^3)^2 + (\gamma_1^3 - \gamma_2^3)^2\right]} = 0$

$\Leftrightarrow (\alpha_1^3 - \alpha_2^3)^2 = 0$, $(\beta_1^3 - \beta_2^3)^2 = 0$, $(\gamma_1^3 - \gamma_2^3)^2 = 0$

$\Leftrightarrow \alpha_1 = \alpha_2, \beta_1 = \beta_2$ and $\gamma_1 = \gamma_2$

Therefore, $\tilde{R}_1 = \tilde{R}_2$

(b) Zhang and Xu distance:

(i) $D^{ZX} = \frac{1}{2}\left(|\alpha_1^3 - \alpha_2^3| + |\beta_1^3 - \beta_2^3| + |\gamma_1^3 - \gamma_2^3|\right)$
$< \frac{1}{2}\left((\alpha_1^3 + \alpha_2^3) + (\beta_1^3 + \beta_2^3) + (\gamma_1^3 + \gamma_2^3)\right)$
$= \frac{1}{2}\left[(\alpha_1^3 + \beta_1^3 + \gamma_1^3) + (\alpha_2^3 + \beta_2^3 + \gamma_2^3)\right]$
$\leq \frac{1}{2}(1+1)$
≤ 1
Therefore, $0 \leq D^{ZX} \leq 1$

Similarly, the other two properties can be proved. □

5. Proposed Raptean Fuzzy MCGDM Framework

This section presents two multi-criteria group decision-making frameworks under Raptean fuzzy environments, one is based on the distance measure technique adopted from the concept of normalized Euclidean distance [39] and normalized Zhang and Xu distance [40], and another one is the Raptean fuzzy WASPAS method. One of the MCDM technique groupings of the weighted sum model (WSM) and the weighted product model (WPM) is the WASPAS approach, which was proposed by Zavadskas et al. [42], provides the basis for ranking the options. An MCGDM technique is used by a group of decision-makers to choose the ideal choice from diverse possibilities by taking into account certain factors.

5.1 Proposed Raptean Fuzzy Distance Measure based MCGDM Framework

The steps of the proposed distance measure-based MCGDM technique under Raptean fuzzy environments are discusses follows:

Step 1: Collect the information needed to address decision-making issues.

Assume that $A = \{A_i | i = 1, 2, \ldots, m\}$ represents the set of "m" potential solutions, and $C = \{C_j | j = 1, 2, \ldots, n\}$ represents the set of "n" criteria appropriate weights is $\theta = \{\theta_j | j = 1, 2, \ldots, n\}$. The group of "$p$" experts (EXs') $E = \{E_k | k = 1, 2, \ldots, p\}$ with appropriate weights $\phi = \{\phi_k | k = 1, 2, \ldots, p\}$.

Table 1
Proposed Linguistic Terms and related RFNs

Linguistic Terms	$\langle \alpha, \beta, \gamma \rangle$
Absolutely More Priority (AMP)	$\langle 0.95, 0.35, 0.20 \rangle$
Very High Priority (VHP)	$\langle 0.90, 0.40, 0.25 \rangle$
High Priority (HP)	$\langle 0.85, 0.45, 0.30 \rangle$
Marginally High Priority (MHP)	$\langle 0.80, 0.50, 0.35 \rangle$
Equal Priority (EP)	$\langle 0.65, 0.65, 0.50 \rangle$
Marginally Low Priority (MLP)	$\langle 0.50, 0.80, 0.35 \rangle$
Low Priority (LP)	$\langle 0.45, 0.85, 0.30 \rangle$
Very Low Priority (VLP)	$\langle 0.40, 0.90, 0.25 \rangle$
Absolutely Low Priority (ALP)	$\langle 0.35, 0.95, 0.20 \rangle$

Step 2: For every criterion, create the decision matrices (DMs).

This step, a set of criteria $C_j (j = 1, 2, \ldots, n)$ will be discussed by a group of EXs, $E_k (k = 1, 2, \ldots, p)$. Convert all of the EXs' thoughts to the RFN linguistic terms listed in Table 1. Consequently, $\mathbf{X} = \left[\tilde{R}_{jk}\right]_{n \times p}$ represents the primary DM of all criterion. Where, $\tilde{R}_{jk} = \langle \alpha_{jk}, \beta_{jk}, \gamma_{jk} \rangle$, $(j = 1, 2, \ldots, n)$ and $(k = 1, 2, \ldots, p)$ are RFNs.

Step 3: Combine the judgments of EXs.

The EXs' ratings are combined at this step for all criteria by using the EXs' weights $\phi_k (k = 1, 2, \ldots, p)$ and the RFWAM operator provided in Eq. (5) or the RFWGM operator from Eq. (6). Next, $\mathbf{X}^{\phi} = \left[\tilde{R}_j^{\phi}\right]_{n \times 1}$ represents the combined RF weights of all the criteria.

Step 4: Defuzzification of aggregated RF weights.

In this step, the score function from Eq. (9) is used to defuzzify the combined RF weights of all the criteria, adopted from the *Definition 3.2*.

$$\mathcal{S}(\tilde{R}_j^{\phi}) = \frac{1 + (\alpha_j^3 - \beta_j^3)(1 - \gamma_j^3)}{2} \in [0, 1]; (j = 1, 2, \ldots, n) \tag{9}$$

Step 5: Figure out the criterion's weights.

This step computes the weights of each criterion given by the Eq. (10) below:

$$\theta_j = \frac{\mathcal{S}(\tilde{R}_j^{\phi})}{\sum_{j=1}^{n} \mathcal{S}(\tilde{R}_j^{\phi})}; (j = 1, 2, \ldots, n) \tag{10}$$

Step 6: Formulate the DMs for each alternative taking into account every criterion.

In this step a group of EXs, $E_k(k = 1, 2, \ldots, p)$ will discuss a collection of alternatives, $A_i(i = 1, 2, \ldots, m)$, in relation to a set of criteria, $C_j(j = 1, 2, \ldots, n)$. Convert all of the EXs' thoughts to the RFN linguistic terms listed in Table 1. Consequently, $E_k = \left[\tilde{R}_{ij}\right]_{m\times n}$ represents the DMs. Where, $\tilde{R}_{ij} = \langle \alpha_{ij}, \beta_{ij}, \gamma_{ij} \rangle$, $(i = 1, 2, \ldots, m)$ and $(j = 1, 2, \ldots, n)$ are RFNs.

Step 7: Combine all of the DMs.

Using the EXs' weights $\phi_k(k = 1, 2, \ldots, p)$ and the RFWAM operator from Eq. (5) or the RFWGM operator from Eq. (6), every DM is combined in this stage, which is denoted by $\mathbf{D}^{\phi} = \left[\tilde{R}_{ij}^{\phi}\right]_{m\times n}$.

Step 8: Group DM normalization.

This step normalizes each element of the group DM according to the beneficiary criterion (BC) and non-beneficiary criterion (NBC), which are provided below and can be computed using Eq. (11);

$$\tilde{R}'_{ij} = \begin{cases} \tilde{R}_{ij}^{\phi}, & \text{for beneficiary criterion} \\ & j = 1, 2, \ldots, n; \forall i \\ (\tilde{R}_{ij}^{\phi})^c, & \text{for non-beneficiary criterion} \end{cases} \tag{11}$$

where, $(\tilde{R}_{ij}^{\phi})^c$ is the complement of $\tilde{R}_{ij}^{\phi}$ which is given in *Definition 3.1*.

Step 9: Compute the Ideal alternatives $(\tilde{I}_j^{+})$ against each criterion based on BC and NBC given by Eq. (12), adopted from the *Definition 3.4*.

$$\tilde{I}_j^{+} = \langle \alpha_j^{+}, \beta_j^{+}, \gamma_j^{+} \rangle = \begin{cases} \max\limits_{1\le i\le m} \tilde{R}'_{ij}, \ j \in BC \\ j = 1, 2, \ldots, n \\ \min\limits_{1\le i\le m} \tilde{R}'_{ij}, \ j \in NBC \end{cases} \tag{12}$$

where,

$$\max_{1\le i\le m} \tilde{R}'_{ij} = \langle max\{\alpha_{ij}\}, min\{\beta_{ij}\}, min\{\gamma_{ij}\} \rangle, \forall j$$
$$\text{and } \min_{1\le i\le m} \tilde{R}'_{ij} = \langle min\{\alpha_{ij}\}, max\{\beta_{ij}\}, max\{\gamma_{ij}\} \rangle, \forall j.$$

Step 10: Compute the weighted distances by considering the calculated weights of criteria $\theta_j(j = 1, 2, \ldots, n)$ given in Eq. (10) and using the Eqs. (13) and (14), adopted from the *Definition 4.3*.

$$D_i^E = \frac{1}{n}\sum_{j=1}^{n} \theta_j \sqrt{\frac{1}{2}\left[\left((\alpha_{ij})^3 - (\alpha_j^{+})^3\right)^2 + \left((\beta_{ij})^3 - (\beta_j^{+})^3\right)^2 + \left((\gamma_{ij})^3 - (\gamma_j^{+})^3\right)^2\right]} \tag{13}$$

$$D_i^{ZX} = \frac{1}{2n}\sum_{j=1}^{n}\theta_j\left(\left|(\alpha_{ij})^3 - (\alpha_j^+)^3\right| + \left|(\beta_{ij})^3 - (\beta_j^+)^3\right| + \left|(\gamma_{ij})^3 - (\gamma_j^+)^3\right|\right) \tag{14}$$

Step 11: Sort the options so that "one" is assigned to the option that is closest to the ideal option.

5.2 Proposed Raptean Fuzzy WASPAS Method for MCGDM Problem

The steps of the introduced Raptean fuzzy WASPAS method-based MCGDM technique are described below:

Steps 1-8: Steps are referred from *Section 5.1*.

Step 9: Determine the weighted DM.

This step computed the weighted DM based on weighted sum model (WSM) and weighted product model (WPM) by considering the calculated weights of criteria $\theta_j (j = 1, 2, \ldots, n)$ given in Eq. (10) and using the RFWAM operator to determine the additive relative importance (S_i) obtained by Eq. (5) and the RFWGM operator to calculated the multiplicative relative importance (P_i) obtained from Eq. (6).

Step 10: Defuzzify the computed additive relative importance (S_i) and multiplicative relative importance (P_i).

Employing the score function provided by Eq. (15), taken from the *Definition 3.2*, and the related score values represented by u_i and v_i, respectively, this step defuzzifies the aggregated S_i and P_i of all the options.

$$\mathcal{S}(\tilde{R}_i^\theta) = \frac{1 + (\alpha_i^3 - \beta_i^3)(1 - \gamma_i^3)}{2} \in [0, 1]; (i = 1, 2, \ldots, m) \tag{15}$$

Step 11: Compute the WASPAS score (W_i) values of alternatives given by Eq. (16).

$$W_i = \mu \cdot u_i + (1 - \mu) \cdot v_i \tag{16}$$

where, $\mu \in [0, 1]$ represented as WASPAS parameter for decision making.

Step 12: Ranking the alternatives.

In this step, all of the options were arranged using score values in descending order by using the *Definition 3.5*.

6. Application: Sustainable Evaluation of Clean Energy Barriers

In this section, we have employed a real-world MCGDM problem to demonstrate our proposed MCGDM strategy under the Raptean fuzzy environments defined in *Section 5.1* and *5.2*. The adopted problem is "clean energy barriers evaluation" [41] with spherical fuzzy information, using an integrated decision-making algorithm tackled with the Failure Mode and Effects Analysis (FMEA) technique. One of the many economic and environmental advantages of clean energy is that it mitigates air pollution. Furthermore, compared to fossil fuels, it may be more environmentally friendly due to its qualities. The rise of sustainable energy is impeded, according to business executives, by high costs, technology limitations, and transmission network constraints. Clean energy barriers are evaluated based on three primary criteria: severity (S), occurrence (O), and detection (D). To prioritize the barriers to the growth of clean energy, a team of three experts (EXs) classified 19 barriers into five groupings: technical, economic, social and cultural, administrative and legal, and political.

6.1 Methodology Implementation

In this subsection, we illustrate Algorithm *5.1* by solving the clean energy barriers evaluation problem adopted from [41].

Step 1: A team of experts, $E = \{E_1, E_2, E_3\}$ with the corresponding weights $\phi = \{0.3, 0.4, 0.3\}$ have been selected 19 alternatives $A = \{B_i | i = 1, 2, \ldots, 19\}$ for clean energy barriers evaluation against a set of three criteria $C = \{S, O, D\}$.

Step 2: The decision matrix for criteria and alternative evaluation has been collected [41], and all the spherical fuzzy LVs have been converted to Raptean fuzzy LVs and RFNs from Table 1.

Step 3-5: Combine the evaluations of each criterion using the EXs weights and the RFWAM operator given by Eq. (5). Defuzzify the aggregated RF weights of each criterion with the score function provided by Eq. (9) and compute the weights of each criterion given by Eq. (10) and presented in Table 2.

Step 6-8: Aggregate the evaluations of each alternative against each criterion utilizing the EXs weights and RFWAM operator given by Eq. (5). Therefore, the normalized group decision matrix is obtained by Eq. (11) according to beneficiary criteria (severity (S) and occurrence (O)) and non-beneficiary criteria (detection (D)), and given in Table 3.

Step 9: Computed the Ideal alternatives $(\tilde{I}_j^+)$ against each criterion based on BC and NBC given by Eq. (12).

Step 10-11: Determining the weighted distances using the computed weights $(0.606, 0.206, 0.187)$ of criteria and using the Eqs. (13) and (14). The alternatives were ranked in ascending order of the distances, i.e., the shortest distance to the ideal alternative considered ranked as 'one', and presented in Table 4.

Table 2
Aggregated weights of criteria

Criteria	RF weights	Score value	Weights
S	$\langle 0.832, 0.469, 0.320\rangle$	0.729	0.606
O	$\langle 0.454, 0.849, 0.306\rangle$	0.248	0.206
D	$\langle 0.436, 0.865, 0.287\rangle$	0.225	0.187

Table 3
Aggregated and Normalized group decisions

Barriers	S	O	D
B1	$\langle 0.799, 0.535, 0.397\rangle$	$\langle 0.817, 0.484, 0.335\rangle$	$\langle 0.469, 0.832, 0.320\rangle$
B2	$\langle 0.776, 0.538, 0.397\rangle$	$\langle 0.787, 0.524, 0.382\rangle$	$\langle 0.457, 0.850, 0.308\rangle$
B3	$\langle 0.822, 0.479, 0.330\rangle$	$\langle 0.759, 0.561, 0.421\rangle$	$\langle 0.464, 0.837, 0.315\rangle$
B4	$\langle 0.800, 0.500, 0.350\rangle$	$\langle 0.709, 0.601, 0.457\rangle$	$\langle 0.500, 0.800, 0.350\rangle$
B5	$\langle 0.805, 0.502, 0.347\rangle$	$\langle 0.710, 0.603, 0.455\rangle$	$\langle 0.502, 0.805, 0.347\rangle$
B6	$\langle 0.799, 0.535, 0.397\rangle$	$\langle 0.822, 0.479, 0.330\rangle$	$\langle 0.464, 0.837, 0.315\rangle$
B7	$\langle 0.900, 0.400, 0.250\rangle$	$\langle 0.909, 0.398, 0.247\rangle$	$\langle 0.419, 0.883, 0.269\rangle$
B8	$\langle 0.931, 0.377, 0.226\rangle$	$\langle 0.919, 0.384, 0.234\rangle$	$\langle 0.400, 0.900, 0.250\rangle$
B9	$\langle 0.900, 0.400, 0.250\rangle$	$\langle 0.906, 0.403, 0.252\rangle$	$\langle 0.419, 0.883, 0.269\rangle$
B10	$\langle 0.815, 0.500, 0.363\rangle$	$\langle 0.711, 0.605, 0.458\rangle$	$\langle 0.485, 0.816, 0.332\rangle$
B11	$\langle 0.811, 0.502, 0.362\rangle$	$\langle 0.709, 0.601, 0.457\rangle$	$\langle 0.484, 0.817, 0.335\rangle$
B12	$\langle 0.650, 0.650, 0.500\rangle$	$\langle 0.557, 0.752, 0.417\rangle$	$\langle 0.484, 0.817, 0.335\rangle$
B13	$\langle 0.800, 0.500, 0.350\rangle$	$\langle 0.817, 0.484, 0.335\rangle$	$\langle 0.464, 0.837, 0.315\rangle$
B14	$\langle 0.832, 0.469, 0.320\rangle$	$\langle 0.919, 0.384, 0.234\rangle$	$\langle 0.434, 0.868, 0.284\rangle$
B15	$\langle 0.800, 0.500, 0.350\rangle$	$\langle 0.878, 0.428, 0.279\rangle$	$\langle 0.484, 0.817, 0.335\rangle$
B16	$\langle 0.832, 0.469, 0.320\rangle$	$\langle 0.811, 0.502, 0.362\rangle$	$\langle 0.538, 0.776, 0.397\rangle$
B17	$\langle 0.868, 0.434, 0.284\rangle$	$\langle 0.873, 0.429, 0.279\rangle$	$\langle 0.400, 0.900, 0.250\rangle$
B18	$\langle 0.868, 0.434, 0.284\rangle$	$\langle 0.887, 0.414, 0.264\rangle$	$\langle 0.419, 0.883, 0.269\rangle$
B19	$\langle 0.900, 0.400, 0.250\rangle$	$\langle 0.924, 0.387, 0.236\rangle$	$\langle 0.393, 0.915, 0.242\rangle$
Ideal value	$\langle 0.931, 0.377, 0.226\rangle$	$\langle 0.924, 0.384, 0.234\rangle$	$\langle 0.393, 0.915, 0.397\rangle$

Table 4
Ranking results by distance measure

Barriers	D^X	Rank	D^{ZX}	Rank
B1	0.06627	13	0.06462	13
B2	0.07375	16	0.07159	16
B3	0.06367	10	0.05982	11
B4	0.07810	18	0.07403	18
B5	0.07636	17	0.07285	17
B6	0.06529	12	0.06372	12
B7	0.01717	3	0.01516	3
B8	0.00321	1	0.00314	1
B9	0.01762	4	0.01558	4
B10	0.07265	14	0.07051	14
B11	0.07374	15	0.07127	15
B12	0.12916	19	0.13289	19
B13	0.06417	11	0.05926	10
B14	0.03994	7	0.03640	7
B15	0.05934	8	0.05447	8
B16	0.06080	9	0.05523	9
B17	0.03130	6	0.02770	6
B18	0.03096	5	0.02762	5
B19	0.01356	2	0.01097	2

Now we illustrate Algorithm *5.2* by solving the clean energy barriers evaluation problem adopted from [41].

Steps 1-8: Steps are presented in Tables 2-3.

Step 9: Determining the weighted decision matrix based on weighted sum model (WSM) and weighted product model (WPM) by considering the calculated weights of criteria $(0.615, 0.202, 0.184)$ using Eq. (5) and Eq. (6).

Step 10: Defuzzifying the additive relative importance (S_i) and multiplicative relative importance (P_i), and their score values represented by u_i and v_i, respectively, determined from the score function from Eq. (15).

Step 11: Computed the WASPAS score (W_i) values of alternatives for the WASPAS parameter $\mu = 0.5$ given by Eq. (16).

Step 12: Ranked all the alternatives in descending order of WASPAS score, and all the results are presented in Table 5.

Table 5
Ranking results by the RF-WASPAS method

Barriers	RF-WSM		RF-WPM		RF-WASPAS Score (W_i)	Rank
	S_i	u_i	P_i	v_i		
B1	$\langle 0.772, 0.569, 0.381 \rangle$	0.630	$\langle 0.727, 0.629, 0.371 \rangle$	0.564	0.597	12
B2	$\langle 0.747, 0.583, 0.388 \rangle$	0.603	$\langle 0.705, 0.645, 0.376 \rangle$	0.539	0.571	18
B3	$\langle 0.776, 0.550, 0.354 \rangle$	0.644	$\langle 0.727, 0.622, 0.352 \rangle$	0.568	0.606	10
B4	$\langle 0.751, 0.567, 0.378 \rangle$	0.614	$\langle 0.715, 0.616, 0.380 \rangle$	0.561	0.588	17
B5	$\langle 0.755, 0.570, 0.375 \rangle$	0.617	$\langle 0.718, 0.619, 0.378 \rangle$	0.563	0.590	15
B6	$\langle 0.773, 0.569, 0.379 \rangle$	0.632	$\langle 0.726, 0.631, 0.370 \rangle$	0.562	0.597	13
B7	$\langle 0.873, 0.463, 0.259 \rangle$	0.778	$\langle 0.782, 0.620, 0.268 \rangle$	0.618	0.698	2
B8	$\langle 0.902, 0.446, 0.240 \rangle$	0.818	$\langle 0.793, 0.631, 0.251 \rangle$	0.621	0.720	1
B9	$\langle 0.872, 0.465, 0.260 \rangle$	0.776	$\langle 0.781, 0.620, 0.269 \rangle$	0.617	0.697	3
B10	$\langle 0.763, 0.570, 0.385 \rangle$	0.622	$\langle 0.719, 0.625, 0.383 \rangle$	0.560	0.591	14
B11	$\langle 0.759, 0.571, 0.384 \rangle$	0.619	$\langle 0.716, 0.625, 0.382 \rangle$	0.558	0.588	16
B12	$\langle 0.609, 0.699, 0.468 \rangle$	0.448	$\langle 0.596, 0.716, 0.453 \rangle$	0.430	0.439	19
B13	$\langle 0.772, 0.547, 0.346 \rangle$	0.642	$\langle 0.726, 0.619, 0.342 \rangle$	0.570	0.606	11
B14	$\langle 0.830, 0.505, 0.301 \rangle$	0.715	$\langle 0.752, 0.623, 0.305 \rangle$	0.589	0.652	6
B15	$\langle 0.793, 0.531, 0.335 \rangle$	0.668	$\langle 0.742, 0.601, 0.340 \rangle$	0.592	0.630	9
B16	$\langle 0.798, 0.523, 0.343 \rangle$	0.675	$\langle 0.763, 0.576, 0.356 \rangle$	0.621	0.648	8
B17	$\langle 0.836, 0.497, 0.287 \rangle$	0.726	$\langle 0.751, 0.645, 0.283 \rangle$	0.576	0.651	7
B18	$\langle 0.841, 0.491, 0.285 \rangle$	0.733	$\langle 0.761, 0.628, 0.286 \rangle$	0.594	0.663	5
B19	$\langle 0.877, 0.464, 0.254 \rangle$	0.783	$\langle 0.775, 0.652, 0.261 \rangle$	0.592	0.687	4

6.2 Sensitivity Analysis

Sensitivity analysis is a helpful technique for assessing the robustness, adaptability, and stability of an MCGDM framework or decision in a variety of situations. The criteria weights are applied as a distinct variable in a decision-making model to carry out the sensibility analysis and verify the effect on the alternatives' ranking. In this instance, sensitivity analysis is carried out by examining the outcomes according to changes in the parameter "μ". The parameter "μ" is crucial to our proposed MCGDM framework because it is utilized in the suggested RF-WASPAS approach to integrate the multiplicative and additive relative importance. Thus, Fig. 3 shows the changes in WASPAS scores when the parameter $\mu \in [0, 1]$ is varied. According to the acquired results, the barrier B_8 has the highest score value, and the barrier B_{12} has the lowest score value for every value between $\mu = 0.0$ and $\mu = 1.0$, which implies "Incomplete markets for energy efficiency" (B_8) is the most appropriate option, while the "Equipment theft" (B_{12}) is the worst option for clean energy barriers evaluation. This exhibits the Raptean fuzzy MCGDM technique's adaptability and stability.

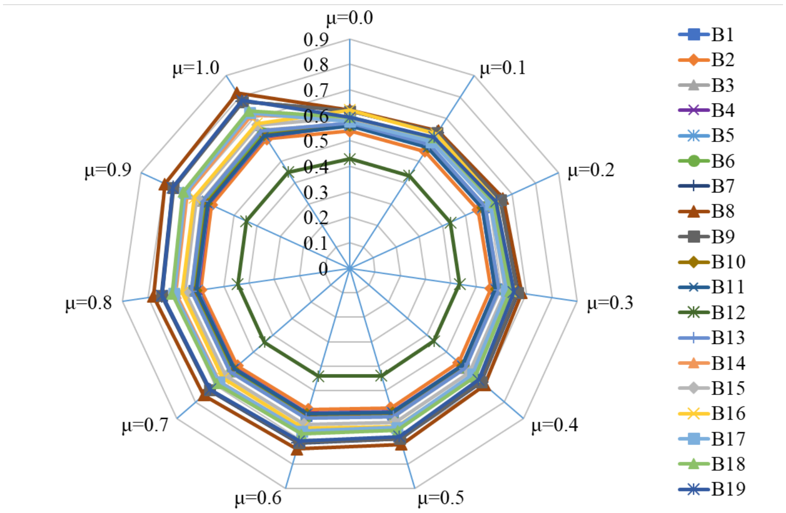


Fig. 3. Variation of score values for the parameter μ

6.3 Comparative Analysis

This subsection presents a comparative analysis based on acquired ranking results from the distance measure technique given in Table 4 and the ranking results from the RF-WASPAS method given in Table 5 with the existing ranking results [41] for the clean energy barriers evaluation. The variation of ranking results is shown in Fig. 4. It is evident that the barrier B_8 has ranked "one" and the barrier B_{12} has ranked "nineteen" for all of the MCGDM approaches under Raptean fuzzy environment, which implies "Incomplete markets for energy efficiency" (B_8) is the most appropriate option, while the "Equipment theft" (B_{12}) is the worst option for clean energy barriers evaluation. Additionally, the Spearman's rank correlation coefficient (SRCC) graph shown in Fig. 5 shows that all of the MCGDM techniques have SRCC values larger than 0.8, indicating a strong correlation between the ranking outcomes. This guarantees the effectiveness and reliability of the new Raptean fuzzy MCGDM frameworks.

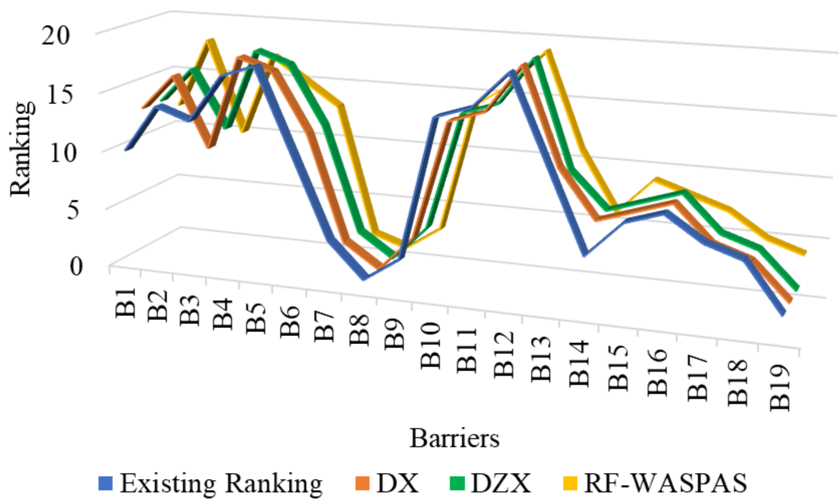


Fig. 4. Ranking variation for different approaches

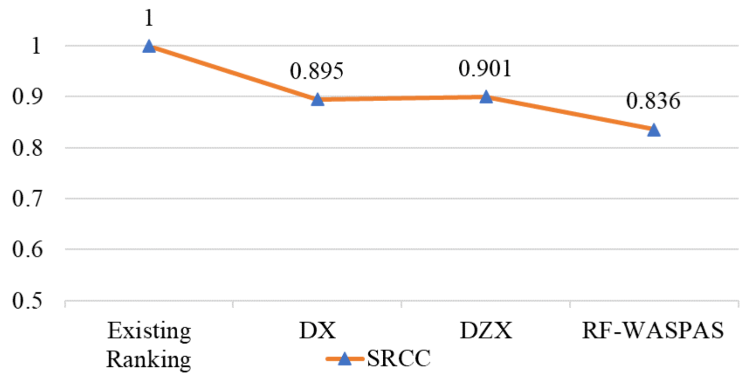


Fig. 5. Correlation between various methods of decision-making

7. Conclusion and Future Research Scope

This research introduces the Raptean fuzzy set (RFS), a direct extension of PiFS, SFS, and FFS concepts. Then, algebraic procedures on RFS with fundamental features are discussed. This article provides an overview of fuzzy set theory. We identify the algebraic operations for the RFS. We establish score and accuracy functions to compare RFSs. Our research focuses on RFS aggregation operators, including rigorous theorems and proofs. In addition, we propose normalized Euclidean distance and Zhang and Xu distance between two RFSs. We introduce the Raptean fuzzy WASPAS approach and a new Raptean distance-based ranking algorithm. Illustrate an MCGDM example, "clean energy barriers evaluation" and solve it with the proposed two Raptean fuzzy MCGDM frameworks. The examined sensitivity analysis for the acquired findings shows the Raptean fuzzy MCGDM technique's adaptability and stability, and ensures the efficiency and trustworthiness of the proposed frameworks. In the future, this study will explore advanced AOs such as Einstein AOs [43], Bonferroni mean operator [44], Heronian mean operator [45]; we will also combine the Raptean fuzzy information with other MCDM methods such as AHP [25], TOPSIS [24], MULTIMOORA [33], CoCoSo [34] with different real-world decision-making applications in different field of study, including medical diagnostics, waste management, location selection, environmental sustainability, transportation and logistics, supply chain management, image processing, cluster analysis, data information technology and many engineering applications.

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Conflicts of Interest

The authors declare no conflicts of interest.

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