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A Generalized Distance Operator and its Applications under Probabilistic Uncertain Linguistic T-Spherical Fuzzy Environment

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ABSTRACT

To address complex decision-making problems under uncertainty, this paper integrates probabilistic uncertain linguistic term sets (PULTS) and T-spherical fuzzy sets (TSFS) to propose a novel model termed probabilistic uncertain linguistic T-spherical fuzzy sets (PULTSFS). This model comprehensively characterizes fuzzy linguistic evaluation information with probability from three dimensions: membership, neutrality, and non-membership. Furthermore, by extending the ordered weighted distance (OWD) measure, the probabilistic uncertain linguistic T-spherical fuzzy ordered weighted distance (PULTSFOWD) operator is defined, and its mathematical properties along with derived forms − PULTSFOWHD, PULTSFOWED and PULTSFOWGD, those are systematically investigated. Based on this operator, a multi-criteria decision-making method is constructed, in which alternatives are ranked by calculating the aggregated distance between each alternative and the positive ideal solution. The proposed method is validated through a case study on green supplier selection, demonstrating its effectiveness and feasibility. Moreover, the PULTSFOWHD and PULTSFOWED operators exhibit relatively stronger robustness under parameter variations, offering a new tool for complex uncertain decision-making problems.

1. Introduction

To overcome the limitations of classical fuzzy sets, which exclusively incorporate membership information [1], Atanassov [2] introduced the concept of intuitionistic fuzzy sets (IFS), enabling the simultaneous consideration of both membership and non-membership information. However, IFS are constrained by the condition that the sum of membership and non-membership degrees cannot exceed 1. To address this limitation, Yager [3] proposed Pythagorean fuzzy sets (PyFS), which relax this constraint by requiring that the sum of the squares of membership and non-membership degrees does not exceed 1, thereby endowing PyFS with a stronger capacity to represent fuzzy information than IFS. Moreover, Yager [4] further generalized this framework by introducing the concept of q-rung orthopair fuzzy sets (q-ROFS), which extend both IFS and PyFS. By allowing the sum of the q-th

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powers of membership and non-membership degrees to be no greater than 1, *q*-ROFS offer a broader capability to represent fuzzy information and enhance operational flexibility. Nevertheless, in certain decision-making contexts, individual evaluations may not be limited to "affirmation" or "negation" but may also encompass varying degrees of "neutrality." To accommodate such situations, Cuong and Kreinovich [5] proposed picture fuzzy sets (PFS), which impose the condition that the sum of membership, neutrality, and non-membership degrees does not exceed 1. Mahmood *et al.,* [6] further extended this framework by introducing spherical fuzzy sets (SFS) and T-spherical fuzzy sets (TSFS), where TSFS satisfy the condition that the sum of the q-th powers of membership, neutrality, and non-membership degrees does not exceed 1 ($q \geq 1$). Compared with PFS and SFS, TSFS provide experts with a more comprehensive and flexible means to express their assessments, granting greater expressive freedom. Consequently, TSFS are recognized as a robust and effective tool for representing and processing fuzzy information [7-10].

The Ordered Weighted Distance (OWD) measure can mitigate (or amplify) the influence of extremely large or small deviations on the aggregation results by assigning lower (or higher) weights to such deviations. Due to this characteristic, OWD has gained considerable favor among scholars and has been generalized into various distance measures and aggregation operators. Merigó and Gil-Lafuente [11] identified two key advantages of the OWAD operator: 1) it can incorporate the attitudinal characteristics of DM. 2) it provides a parameterized family of distance aggregation operators ranging from the minimum to the maximum distance. The OWD measure and the OWAD operator have been extended and applied by researchers in diverse decision-making contexts. For example, Zeng and Su [12] integrated OWD with IFS to propose a series of IFOWD operators; Şahin and Küçük [13] developed a family of SNOWD operators in the context of single-valued neutrosophic sets; Zeng *et al.,* [14] introduced the Pythagorean Fuzzy Ordered Weighted Averaging Weighted Averaging Distance (PyFOWAWAD) operator in a Pythagorean fuzzy environment; Su *et al.,* [15] constructed a series of OWAD aggregation operators under Atanassov's intuitionistic linguistic term sets; and Liu *et al.,* [16] extended the OWD measure to the probabilistic linguistic term set setting and applied it to the evaluation of public satisfaction with ecological environments. However, none of these distance aggregation operators are applicable to the domain of the PULTSFS.

Therefore, the main objective of this paper is to extend the aforementioned ordered weighted measures by generalizing the OWD measure and the OWAD operator to the context of (PULTSFS). Based on this aim, the paper proposes a Probabilistic Uncertain Linguistic T-Spherical Fuzzy Ordered Weighted Distance (PULTSFOWD) operator along with several of its variants. Using these distance aggregation operators, DM can simply select an appropriate operator that aligns with their specific focus or scenario.

The structure of the paper is organized as follows: Section 2 introduces the relevant fuzzy set theories and the ordered weighted distance operator. Section 3 defines the new concept of PULTSFS. Section 4 develops the PULTSFOWD aggregation operator and elaborates its fundamental properties. Section 5 discusses the family of PULTSFOWD operators. Section 6 constructs a decision-making method for PULTSFS based on distance aggregation operators. Section 7 provides an example to demonstrate the effectiveness and superiority of the proposed method. Finally, Section 8 summarizes the contributions of this study.

2. Preliminaries

2.1 PULTS, TSFS

To more accurately express the concerns of decision makers, Lin *et al.,* [17] proposed the concept of PULTS based on ULV [18] and PLTS [19].

Definition 1 [17]. A PULTS is defined as follows:

$$UL(p)=\left\{\langle[s_\alpha^{(t)},s_\beta^{(t)}],p^{(t)}\rangle \mid p^{(t)}\geq 0,t=1,2,...,\#T,\ \sum_{t=1}^{\#T} p^{(t)}\leq 1\right\} \tag{1}$$

Among them, $\langle[s_\alpha^{(t)},s_\beta^{(t)}],p^{(t)}\rangle$ is the t-th PUL element in $UL(p)$, $s_\alpha^{(t)}$, $s_\beta^{(t)}$ are the lower and upper language terms, respectively, and $p^{(t)}$ is the corresponding probability.

Definition 2 [17]. Let $UL_1(p)=\{\langle[s_{\alpha 1}^{(t)}, s_{\beta 1}^{(t)}], p_1^{(t)}\rangle | p_1^{(t)}\geq 0,t=1,2,...,\#T_1\}$ and $UL_2(p)=\{\langle[s_{\alpha 2}^{(t)}, s_{\beta 2}^{(t)}], p_2^{(t)}\rangle | p_2^{(t)}\geq 0\ ,t=1,2,...,\#T_2\}$ be two PULTSs. For ease of calculation, they need to be normalized as follows :

(1) If $0<\sum_{t=1}^{\#T_j} p_j^{(t)}<1$, then $UL_j(p)$ is normalized to $UL_j(\tilde{p})=\left\{\langle[s_{\alpha j}^{(t)}, s_{\beta j}^{(t)}], \tilde{p}_j^{(t)}\rangle \,\middle|\, \tilde{p}_j^{(t)}\geq 0,t=1,2,...,\#T_j, \sum_{t=1}^{\#T_j} \tilde{p}_j^{(t)}=1\right\}$, where $\tilde{p}_j^{(t)}= p_j^{(t)}/\sum_{t=1}^{T_j} p_j^{(t)}$, j=1, 2.

(2) If $\#T_1\neq\#T_2$, then we need to add some linguistic terms to the one with fewer elements, the probability is 0, and the normalized PULTS is obtained.

Compared with traditional fuzzy sets and their variants, TSFS can provide experts with a broader expression space and greater flexibility in three-dimensional aspects. Mahmood *et al.,* [6] gave the following definition:

Definition 3 [6]. Assume that X={x1, x2, ..., xn} is a domain, then a TSFS A on X is an object in the following form:

$$A=\{\langle x_j,(\mu_A(x_j),\eta_A(x_j),v_A(x_j))\rangle | x_j\in X\} \tag{2}$$

Among them, $\mu_A(x_j),\eta_A(x_j),v_A(x_j):X\to[0,1]$ represent the membership, neutrality and non-membership of x_j to A, respectively, and they satisfy the $0\leq(\mu_A(x_j))^q+(\eta_A(x_j))^q+(v_A(x_j))^q\leq 1(q\geq 1)$ condition. In order to facilitate the calculation, a triple $a_j=(\mu_j,\eta_j,v_j)$ is called T-spherical fuzzy number (TSFN), and the parameter q value can be set appropriately. When the q value is larger, the strength of the commitment is smaller, the smaller the q value, the less hesitation, and the less uncertainty.

Definition 4 [20]. Let $a_j=(\mu_j,\eta_j,v_j)$(j=0,1,2) be three TSFNs, λ>0, then their algorithms are as follows:

(1) $a_1\oplus a_2=\left(\sqrt[q]{\mu_1^q+\mu_2^q-\mu_1^q\mu_2^q},\eta_1\eta_2,v_1v_2\right)$;

(2) $a_1\otimes a_2=\left(\mu_1\mu_2,\sqrt[q]{\eta_1^q+\eta_2^q-\eta_1^q\eta_2^q},\sqrt[q]{v_1^q+v_2^q-v_1^qv_2^q}\right)$;

(3) $\lambda a_0=\left(\sqrt[q]{1-(1-\mu_0^q)^\lambda},\eta_0^\lambda,v_0^\lambda\right)$;

(4) $(a_0)^\lambda=\left(\mu_0^\lambda,\sqrt[q]{1-(1-\eta_0^q)^\lambda},\sqrt[q]{1-(1-v_0^q)^\lambda}\right)$;

(5) $(a_0)^c=(v_0,\eta_0,\mu_0)$.

2.2 The OWD operator

Based on the concept of the OWA operator, Xu and Chen [21] proposed the Ordered Weighted Distance (OWD) operator, which is defined as follows:

Definition 5 [21]. Let $a=\{\alpha_1,\alpha_2,...,\alpha_n\}$ and $b=\{\beta_1,\beta_2,...,\beta_n\}$, be two sets of real numbers. For any two elements α_j and β_j the distance $d(\alpha_j,\beta_j)=|\alpha_j-\beta_j|$ is defined. Then,

$$OWD_{(\boldsymbol{a},\boldsymbol{b})}=\left(\sum_{j=1}^n w_j\left(d(\alpha_{\sigma(j)},\beta_{\sigma(j)})\right)^\lambda\right)^{1/\lambda} \tag{3}$$

It is called the Ordered Weighted Distance (OWD) measure between sets a and b. Let λ>0，$(\sigma(1),\sigma(2),\ldots,\sigma(n))$be a permutation of (1,2,...,n), such that$d(\alpha_{\sigma(j-1)},\beta_{\sigma(j-1)}) \geq d(\alpha_{\sigma(j)},\beta_{\sigma(j)})$. $w=(w_1,w_2,\ldots,w_n)^T$ be the weight vector associated with OWD, satisfying 0∈[0,1] and $\sum_{j=1}^{n} w_j = 1$.

In particular, when λ=1, λ=2 and λ→0, the OWD measure is respectively referred to as the Ordered Weighted Averaging Distance (OWAD) operator, the Ordered Weighted Euclidean Distance (OWED) operator, and the Ordered Weighted Geometric Distance (OWGD) operator.

3. The Concept of PULTSFS

3.1 PULTSFS

Based on the advantages of PULTS and TSFS, we define a new type of probabilistic fuzzy set, PULTSFS, which not only allows experts to use multiple language items from three dimensions to express evaluation information, but also includes the possibility of each language item. The definition of PULTSFS is as follows:

Definition 6. Let X be a domain and $S_{[0,\,k\text{-}1]}$ be an LTS. Then define a PULTSFS $\widetilde{A}(p)$ on X as:

$$\widetilde{A}(p)=\left\{\langle x_\varsigma,\ \varphi_\varsigma(\hat{p})(x_\varsigma),\ \phi_\varsigma(\tilde{p})(x_\varsigma),\ \psi_\varsigma(\bar{p})(x_\varsigma)\rangle \middle| x_\varsigma \in X\right\} \tag{4}$$

where $\varphi_\varsigma(\hat{p})(x_\varsigma)=\left\{\left[s_{\mu^L_{\varsigma(t)}},\ s_{\mu^U_{\varsigma(t)}}\right](\hat{p}_{(t)}) \middle| s_{\mu^L_{\varsigma(t)}},\ s_{\mu^U_{\varsigma(t)}} \in S_{[0,\,k\text{-}1]},\ \hat{p}_{(t)}\geq 0,\ \sum_{t=1}^{\#T}\hat{p}_{(t)}\leq 1\right\}$ denotes the membership degree of $x_\varsigma \in X$; $\phi_\varsigma(\tilde{p})(x_\varsigma)=\left\{\left[s_{\eta^L_{\varsigma(r)}},\ s_{\eta^U_{\varsigma(r)}}\right](\tilde{p}_{(r)}) \middle| s_{\eta^L_{\varsigma(r)}},\ s_{\eta^U_{\varsigma(r)}} \in S_{[0,\,k\text{-}1]},\ \tilde{p}_{(r)}\geq 0,\ \sum_{r=1}^{\#R}\tilde{p}_{(t)}\leq 1\right\}$ denotes the neutrality degree of $x_\varsigma \in X$; $\psi_\varsigma(\bar{p})(x_\varsigma)=\left\{\left[s_{\nu^L_{\varsigma(w)}},\ s_{\nu^U_{\varsigma(w)}}\right](\bar{p}_{(w)}) \middle| s_{\nu^L_{\varsigma(w)}},\ s_{\nu^U_{\varsigma(w)}} \in S_{[0,\,k\text{-}1]},\ \bar{p}_{(w)}\geq 0,\ \sum_{w=1}^{\#W}\bar{p}_{(w)}\leq 1\right\}$ denotes the non-membership degree of $x_\varsigma \in X$, and the related probabilities are $\hat{p}_{(t)}$, $\tilde{p}_{(r)}$ and $\bar{p}_{(w)}$, respectively. For $x_\varsigma \in X$, they satisfy the condition $0\leq\left(max_{t=1}^{\#T}\mu^U_{\varsigma(t)}\right)^q+\left(max_{r=1}^{\#R}\eta^U_{\varsigma(t)}\right)^q+\left(max_{w=1}^{\#W}\nu^U_{\varsigma(w)}\right)^q\leq k^q$ (q≥1).

If X = { x }, then PULTSFS degenerates into a PULTSFN, that is,

$$\tilde{a}(p)=\langle\left\{\left[s_{\mu^L_{(t)}},\ s_{\mu^U_{(t)}}\right]\middle|\hat{p}_{(t)}\right\},\left\{\left[s_{\eta^L_{(r)}},\ s_{\eta^U_{(r)}}\right]\middle|\tilde{p}_{(r)}\right\},\left\{\left[s_{\nu^L_{(w)}},\ s_{\nu^U_{(w)}}\right]\middle|\bar{p}_{(w)}\right\}\rangle \tag{5}$$

Where $s_{\mu^L_{(t)}},\ s_{\mu^U_{(t)}},\ s_{\eta^L_{(r)}},\ s_{\eta^U_{(r)}},\ s_{\nu^L_{(w)}},\ s_{\nu^U_{(w)}} \in S_{[0,\,k\text{-}1]}, \sum_{t=1}^{\#T}\hat{p}_{(t)}\leq 1, \sum_{r=1}^{\#R}\tilde{p}_{(t)}\leq 1, \sum_{w=1}^{\#W}\bar{p}_{(w)}\leq 1$.

Example 1. There is an LTS $S_{[0,\,6]}$. Suppose two experts are invited to assess whether a company's potential value is worth investing in. The first expert believes that 30% of the investment is worth expressing opinions in the ULV [s_3, s_4], and 15 % of the neutral attitude is expressed in the ULV [s_2, s_4], and 40 % of the investment is not worth investing. Convinced and express opinions in the ULV [s_1, s_2]. Then the expert's evaluation information can be expressed as: $\tilde{a}_1(p)=\langle\{[s_3,s_4]|0.3\},\{[s_2,s_4]|0.15\},\{[s_1,s_2]|0.4\}\rangle$.

The second expert has some hesitation. He believes that 60 % of the determination is worth investing in the ULV [s_2, s_3], and another 30 % is worth investing in the ULV [s_2, s_4]. At the same time, he holds a 35 % neutral attitude to express his opinion in the ULV [s_3, s_5], and 50 % is not worth investing in the ULV [s_3, s_4]. Then the expert's evaluation information can be expressed as: $\tilde{a}_2(p)=\langle\{[s_2,s_3]|0.6,\ [s_2,s_4]|0.3,\ \},\{[s_3,s_5]|0.35\},\{[s_3,s_4]|0.5\}\rangle$.

Remark 1. For different values of parameter q, PULTSFS can be degraded into different specific forms, as follows :

(1) When q = 1, $\widetilde{A}(p)$ is reduced to probabilistic uncertain linguistic PFS (PULPFS);
(2) When q = 2, $\widetilde{A}(p)$ is reduced to a probabilistic uncertain linguistic SFS (PULSFS);
(3) When$\phi_\varsigma(\tilde{p})(x_\varsigma)$=0, $\widetilde{A}(p)$ is reduced to a PULq-ROFS [22];
(4) When q=1, $\phi_\varsigma(\tilde{p})(x_\varsigma)$=0, $\widetilde{A}(p)$ is reduced to a PULIFS [23];
(5) When q=2, $\phi_\varsigma(\tilde{p})(x_\varsigma)$=0, $\widetilde{A}(p)$ is reduced to a probabilistic uncertain PyFS (PULPyFS);

(6) When $s_{\mu^L_{(t)}}$=$s_{\mu^U_{(t)}}$,$s_{\eta^L_{(r)}}$=$s_{\eta^U_{(r)}}$ and $s_{\nu^L_{(w)}}$=$s_{\nu^U_{(w)}}$, $\widetilde{A}(p)$ is reduced to a PLTSFS [15];

(7) When $\phi_\varsigma(\tilde{p})(x_\varsigma)$=0, $s_{\mu^L_{(t)}}$=$s_{\mu^U_{(t)}}$ *and* $s_{\nu^L_{(w)}}$=$s_{\nu^U_{(w)}}$, $\widetilde{A}(p)$ is reduced to a PL*q*-ROFS [16,17];

(8) When q=1, $\phi_\varsigma(\tilde{p})(x_\varsigma)$=0 , $s_{\mu^L_{(t)}}$=$s_{\mu^U_{(t)}}$ and $s_{\nu^L_{(w)}}$=$s_{\nu^U_{(w)}}$, $\widetilde{A}(p)$ is reduced to a probabilistic linguistic IFS (PLIFS) ;

(9) When q=2, $\phi_\varsigma(\tilde{p})(x_\varsigma)$=0 , $s_{\mu^L_{(t)}}$=$s_{\mu^U_{(t)}}$ and $s_{\nu^L_{(w)}}$=$s_{\nu^U_{(w)}}$, $\widetilde{A}(p)$ is reduced to a probabilistic linguistic PyFS (PLPyFS).

Obviously, PULTSFS has a strong generalization, and in some special cases, PULTSFS can be degraded to a special form.

Definition 7. Let any two PULTSFNs, $\tilde{\alpha}_1(p) = \left\langle\left\{\left[s_{\mu^L_{1(t)}}, s_{\mu^U_{1(t)}}\right]\middle|\hat{p}_{1(t)}\right\},\left\{\left[s_{\eta^L_{1(r)}}, s_{\eta^U_{1(r)}}\right]\middle|\tilde{p}_{1(r)}\right\},\left\{\left[s_{\nu^L_{1(w)}}, s_{\nu^U_{1(w)}}\right]\middle|\bar{p}_{1(w)}\right\}\right\rangle$, $\tilde{\alpha}_2(p) = \langle\left\{\left[s_{\mu^L_{2(t)}}, s_{\mu^U_{2(t)}}\right]\middle|\hat{p}_{2(t)}\right\},\left\{\left[s_{\eta^L_{2(r)}}, s_{\eta^U_{2(r)}}\right]\middle|\tilde{p}_{2(r)}\right\},\left\{\left[s_{\nu^L_{2(w)}}, s_{\nu^U_{2(w)}}\right]\middle|\bar{p}_{2(w)}\right\}\rangle$. For ease of calculation, they need to be normalized as follows:

(1) (Probability normalization) If $0 < \sum_{t=1}^{\#T}\hat{p}_{j(t)} < 1$ (take membership probability as an example, j=1,2), then $\tilde{\alpha}_j(p)$ is normalized to $\tilde{\alpha}_j(p^n)$, then the probability is $\hat{p}^n_{j(t)} = \hat{p}_{j(t)}/\sum_{t=1}^{\#T}\hat{p}_{j(t)}$, and the corresponding normalized PULTSFN can be described as:

$$\tilde{\alpha}_j(p^n) = \langle\left\{\left[s_{\mu^L_{j(t)}}, s_{\mu^U_{j(t)}}\right]\middle|\hat{p}^n_{j(t)}\right\},\left\{\left[s_{\eta^L_{j(r)}}, s_{\eta^U_{j(r)}}\right]\middle|\tilde{p}^n_{j(r)}\right\},\left\{\left[s_{\nu^L_{j(w)}}, s_{\nu^U_{j(w)}}\right]\middle|\bar{p}^n_{j(w)}\right\}\rangle \tag{6}$$

(2)(Structural standardization) If (for example, membership), then you need to add some linguistic terms to the smaller number of elements with fewer terms, assigning them a probability of 0, resulting in normalized PULTSFNs, (j = 1,2).

Example 2. Assume that there are two PULTSFNs, a and b, namely:

$$\tilde{\alpha}_1(p) = \begin{pmatrix}\{[s_3, s_4]|0.2,[s_4, s_5]|0.6\},\\ \{[s_1, s_2]|0.4,[s_2, s_4]|0.6\},\\ \{[s_5, s_6]|0.3,[s_4, s_5]|0.7\}\end{pmatrix},\ \tilde{\alpha}_2(p) = \begin{pmatrix}\{[s_2, s_3]|0.4,[s_3, s_4]|0.2, [s_5, s_6]|0.2\},\\ \{[s_4, s_5]|0.3,[s_2, s_3]|0.6\},\\ \{[s_1, s_2]|0.3,[s_2, s_3]|0.4\}\end{pmatrix}.$$

Then, (1) according to Definition 9, we perform initial standardization on this:

$$\tilde{\alpha}_1(p^n) = \begin{pmatrix}\{[s_3, s_4]|0.25,[s_4, s_5]|0.75\},\\ \{[s_1, s_2]|0.40,[s_2, s_4]|0.60\},\\ \{[s_4, s_5]|0.70,[s_5, s_6]|0.30\}\end{pmatrix},$$

$$\tilde{\alpha}_2(p^n) = \begin{pmatrix}\{[s_2, s_3]|0.50,[s_3, s_4]|0.25, [s_5, s_6]|0.25\},\\ \{[s_4, s_5]|0.33,[s_2, s_3]|0.67\},\\ \{[s_1, s_2]|0.43,[s_2, s_3]|0.57\}\end{pmatrix}$$

$$\tilde{\alpha}_1(p^N) = \begin{pmatrix}\{[s_3, s_4]|0.00, [s_3, s_4]|0.25,[s_4, s_5]|0.75\},\\ \{[s_1, s_2]|0.40,[s_2, s_3]|0.30, [s_3, s_4]|0.30\},\\ \{[s_4, s_5]|0.7 ,[s_5, s_6]|0.3 \}\end{pmatrix},$$

$$\tilde{\alpha}_2(p^N) = \begin{pmatrix}\{[s_2, s_3]|0.50,[s_3, s_4]|0.25, [s_5, s_6]|0.25\},\\ \{[s_2, s_3]|0.00, [s_2, s_3]|0.67, [s_4, s_5]|0.33\},\\ \{[s_1, s_2]|0.43,[s_2, s_3]|0.57\}\end{pmatrix}$$

Definition 8. Assume that $S_{[0,k-1]}$ is an LTS, for any PULTSFN $\tilde{\alpha}(p) = \langle\left\{\left[s_{\mu^L_{(t)}}, s_{\mu^U_{(t)}}\right]\middle|\hat{p}_{(t)}\right\},\left\{\left[s_{\eta^L_{(r)}}, s_{\eta^U_{(r)}}\right]\middle|\tilde{p}_{(r)}\right\},\left\{\left[s_{\nu^L_{(w)}}, s_{\nu^U_{(w)}}\right]\middle|\bar{p}_{(w)}\right\}\rangle$, where $s_{\mu^L_{(t)}}, s_{\mu^U_{(t)}}, s_{\eta^L_{(r)}}, s_{\eta^U_{(r)}}, s_{\nu^L_{(w)}}, s_{\nu^U_{(w)}} \in S_{[0,k-1]}$, ($t$=1,2,...,#$T$; r=1,2,...,#R; w=1,2,...,#W), the score function of $\tilde{\alpha}(p)$ is defined as:

$$Sc(\tilde{a}(p)) = s_{\frac{1}{2}\left(1+\left(\frac{\sum_{t=1}^{\#T}\left(\frac{1}{2}\left(\mu_{(t)}^{L}+\mu_{(t)}^{U}\right)\hat{p}_{(t)}\right)}{k\sum_{t=1}^{\#T}\hat{p}_{(t)}}\right)^{q}-\left(\frac{\sum_{r=1}^{\#R}\left(\frac{1}{2}\left(\eta_{(t)}^{L}+\eta_{(t)}^{U}\right)\tilde{p}_{(r)}\right)}{k\sum_{r=1}^{\#R}\tilde{p}_{(r)}}\right)^{q}-\left(\frac{\sum_{w=1}^{\#W}\left(\frac{1}{2}\left(\nu_{(t)}^{L}+\nu_{(t)}^{U}\right)\bar{p}_{(w)}\right)}{k\sum_{w=1}^{\#W}\bar{p}_{(w)}}\right)^{q}\right)} \tag{7}$$

The accuracy function of $\tilde{a}(p)$ is defined as:

$$Ac(\tilde{a}(p)) = s_{\left(\frac{\sum_{t=1}^{\#T}\left(\frac{1}{2}\left(\mu_{(t)}^{L}+\mu_{(t)}^{U}\right)\hat{p}_{(t)}\right)}{k\sum_{t=1}^{\#T}\hat{p}_{(t)}}\right)^{q}+\left(\frac{\sum_{r=1}^{\#R}\left(\frac{1}{2}\left(\eta_{(t)}^{L}+\eta_{(t)}^{U}\right)\tilde{p}_{(r)}\right)}{k\sum_{r=1}^{\#R}\tilde{p}_{(r)}}\right)^{q}+\left(\frac{\sum_{w=1}^{\#W}\left(\frac{1}{2}\left(\nu_{(t)}^{L}+\nu_{(t)}^{U}\right)\bar{p}_{(w)}\right)}{k\sum_{w=1}^{\#W}\bar{p}_{(w)}}\right)^{q}} \tag{8}$$

Definition 9. Assume that there are two arbitrary sum of PULTSFNs, then their comparison rules are as follows:

（1）If $Sc(\tilde{a}_1(p)) \succ Sc(\tilde{a}_2(p))$, then $\tilde{a}_1(p) \succ \tilde{a}_2(p)$;
（2）If $Sc(\tilde{a}_1(p)) \prec Sc(\tilde{a}_2(p))$, then $\tilde{a}_1(p) \prec \tilde{a}_2(p)$;
（3）If $Sc(\tilde{a}_1(p)) = Sc(\tilde{a}_2(p))$, then,
i) If $Ac(\tilde{a}_1(p)) \succ Ac(\tilde{a}_2(p))$, then $\tilde{a}_1(p) \succ \tilde{a}_2(p)$;
ii) If $Ac(\tilde{a}_1(p)) \prec Ac(\tilde{a}_2(p))$,then $\tilde{a}_1(p) \prec \tilde{a}_2(p)$;
iii) If $Ac(\tilde{a}_1(p)) = Ac(\tilde{a}_2(p))$， then $\tilde{a}_1(p) \approx \tilde{a}_2(p)$;

Example 3. Taking two PULTSFNs $\tilde{a}_1(p)$ and $\tilde{a}_2(p)$ in Example 2 as examples, their score function values are calculated and compared. According to Eq. (7), *q* = 4, then

$$Sc(\tilde{a}_1(p)) = s_{\frac{1}{2}\left(\begin{array}{c}1+\left(\left(\frac{1}{2}\times(3+4)\times0.2+\frac{1}{2}\times(3+4)\times0.6\right)/7\times(0.2+0.6)\right)^{4}\\ -\left(\left(\frac{1}{2}\times(1+2)\times0.4+\frac{1}{2}\times(2+4)\times0.6\right)/7\times(0.4+0.6)\right)^{4}\\ -\left(\left(\frac{1}{2}\times(5+6)\times0.3+\frac{1}{2}\times(4+5)\times0.7\right)/7\times(0.3+0.7)\right)^{4}\end{array}\right)} = s_{0.450}$$

$$Sc(\tilde{a}_2(p)) = s_{\frac{1}{2}\left(\begin{array}{c}1+\left(\left(\frac{1}{2}\times(2+3)\times0.4+\frac{1}{2}\times(3+4)\times0.2+\frac{1}{2}\times(5+6)\times0.2\right)/7\times(0.4+0.2+0.2)\right)^{4}\\ -\left(\left(\frac{1}{2}\times(4+5)\times0.3+\frac{1}{2}\times(2+3)\times0.6\right)/7\times(0.3+0.6)\right)^{4}\\ -\left(\left(\frac{1}{2}\times(1+2)\times0.3+\frac{1}{2}\times(2+3)\times0.4\right)/7\times(0.3+0.4)\right)^{4}\end{array}\right)} = s_{0.506}$$

We have $Sc(\tilde{a}_1(p)) \prec Sc(\tilde{a}_2(p))$, according to the Definition 11, then there is $\tilde{a}_1(p) \prec \tilde{a}_2(p)$.

3.2 The PULTSF distance

To measure the distance between any two PULTSFNs $\tilde{a}_1(p) = \langle\left\{\left[s_{\mu_{1(t)}^{L}}, s_{\mu_{1(t)}^{U}}\right]\middle|\hat{p}_{1(t)}\right\}, \left\{\left[s_{\eta_{1(r)}^{L}}, s_{\eta_{1(r)}^{U}}\right]\middle|\tilde{p}_{1(r)}\right\}, \left\{\left[s_{\nu_{1(w)}^{L}}, s_{\nu_{1(w)}^{U}}\right]\middle|\bar{p}_{1(w)}\right\}\rangle$ and $\tilde{a}_2(p) = \langle\left\{\left[s_{\mu_{2(t)}^{L}}, s_{\mu_{2(t)}^{U}}\right]\middle|\hat{p}_{2(t)}\right\}, \left\{\left[s_{\eta_{2(r)}^{L}}, s_{\eta_{2(r)}^{U}}\right]\middle|\tilde{p}_{2(r)}\right\}, \left\{\left[s_{\nu_{2(w)}^{L}}, s_{\nu_{2(w)}^{U}}\right]\middle|\bar{p}_{2(w)}\right\}\rangle$,the following distance definition is given.

Definition 10. Assume that $S_{[0,k-1]}$ is an LTS, for any two PULTSFNs $\tilde{a}_1(p) = \langle\left\{\left[s_{\mu_{1(t)}^{L}}, s_{\mu_{1(t)}^{U}}\right]\middle|\hat{p}_{1(t)}\right\}, \left\{\left[s_{\eta_{1(r)}^{L}}, s_{\eta_{1(r)}^{U}}\right]\middle|\tilde{p}_{1(r)}\right\}, \left\{\left[s_{\nu_{1(w)}^{L}}, s_{\nu_{1(w)}^{U}}\right]\middle|\bar{p}_{1(w)}\right\}\rangle$ and $\tilde{a}_2(p) = \langle\left\{\left[s_{\mu_{2(t)}^{L}}, s_{\mu_{2(t)}^{U}}\right]\middle|\hat{p}_{2(t)}\right\}, \left\{\left[s_{\eta_{2(r)}^{L}}, s_{\eta_{2(r)}^{U}}\right]\middle|\tilde{p}_{2(r)}\right\}, \left\{\left[s_{\nu_{2(w)}^{L}}, s_{\nu_{2(w)}^{U}}\right]\middle|\bar{p}_{2(w)}\right\}\rangle$, where $\#T_1=\#T_2=\#T$, $\#R_1=\#R_2=\#R$, $\#W_1=\#W_2=\#W$, and for the part of ULVs, function *I* (˙) satisfies $I(s_\alpha) = \alpha$, then the distance $D(\tilde{a}_1(p), \tilde{a}_2(p))$ between $\tilde{a}_1(p)$ and $\tilde{a}_2(p)$ is defined as :

$$D(\tilde{a}_1(p),\tilde{a}_2(p))=\frac{1}{6}\left(\begin{array}{c}\dfrac{\sum_{t=1}^{\#T}\left(\begin{array}{c}\left|\left(I(s_{\mu_{1(t)}^L})\hat{p}_{1(t)}\right)^q-\left(I(s_{\mu_{2(t)}^L})\hat{p}_{2(t)}\right)^q\right|\\+\left|\left(I(s_{\mu_{1(t)}^U})\hat{p}_{1(t)}\right)^q-\left(I(s_{\mu_{2(t)}^U})\hat{p}_{2(t)}\right)^q\right|\end{array}\right)}{\#T}+\\ \dfrac{\sum_{r=1}^{\#R}\left(\begin{array}{c}\left|\left(I(s_{\eta_{1(r)}^L})\tilde{p}_{1(r)}\right)^q-\left(I(s_{\eta_{2(r)}^L})\tilde{p}_{2(r)}\right)^q\right|\\+\left|\left(I(s_{\eta_{1(r)}^U})\tilde{p}_{1(r)}\right)^q-\left(I(s_{\eta_{2(r)}^U})\tilde{p}_{2(r)}\right)^q\right|\end{array}\right)}{\#R}+\\ \dfrac{\sum_{w=1}^{\#W}\left(\begin{array}{c}\left|\left(I(s_{\nu_{1(w)}^L})\bar{p}_{1(w)}\right)^q-\left(I(s_{\nu_{2(w)}^L})\bar{p}_{2(w)}\right)^q\right|\\+\left|\left(I(s_{\nu_{1(w)}^U})\bar{p}_{1(w)}\right)^q-\left(I(s_{\nu_{2(w)}^U})\bar{p}_{2(w)}\right)^q\right|\end{array}\right)}{\#W}\end{array}\right) \tag{9}$$

The PULTSF distance satisfies the properties of non-negativity, symmetry, reflexivity, and the triangle inequality. These properties can be proved through the following theorems.

Theorem 1. For any three PULTSFNs $\tilde{a}(p)=\langle\left\{\left[s_{\mu_{(t)}^L},s_{\mu_{(t)}^U}\right]|\hat{p}_{(t)}\right\},\left\{\left[s_{\eta_{(r)}^L},s_{\eta_{(r)}^U}\right]|\tilde{p}_{(r)}\right\},\left\{\left[s_{\nu_{(w)}^L},s_{\nu_{(w)}^U}\right]|\bar{p}_{(w)}\right\}\rangle$, $\tilde{a}_1(p)=\langle\left\{\left[s_{\mu_{1(t)}^L},s_{\mu_{1(t)}^U}\right]|\hat{p}_{1(t)}\right\},\left\{\left[s_{\eta_{1(r)}^L},s_{\eta_{1(r)}^U}\right]|\tilde{p}_{1(r)}\right\},\left\{\left[s_{\nu_{1(w)}^L},s_{\nu_{1(w)}^U}\right]|\bar{p}_{1(w)}\right\}\rangle$ and $\tilde{a}_2(p)=\langle\left\{\left[s_{\mu_{2(t)}^L},s_{\mu_{2(t)}^U}\right]|\hat{p}_{2(t)}\right\},\left\{\left[s_{\eta_{2(r)}^L},s_{\eta_{2(r)}^U}\right]|\tilde{p}_{2(r)}\right\},\left\{\left[s_{\nu_{2(w)}^L},s_{\nu_{2(w)}^U}\right]|\bar{p}_{2(w)}\right\}\rangle$, then

(1) Non-negativity: $D(\tilde{a}_1(p),\tilde{a}_2(p))\geq 0$;

(2) Commutativity: $D(\tilde{a}_1(p),\tilde{a}_2(p))=D(\tilde{a}_2(p),\tilde{a}_1(p))$;

(3) Reflexivity: $D(\tilde{a}(p),\tilde{a}(p))=0$;

Proof: The proofs for properties (1-3) are straightforward and are omitted here.

Example 4. Taking two PULTSFNs $\tilde{a}_j(p^N)$(j=1, 2) in Example 2 as an example, using the Eq. (9) to calculate the distance between them, q = 4, then we have:

$$D(\tilde{a}_1(p),\tilde{a}_2(p))=\frac{1}{6}\left(\begin{array}{c}\frac{1}{3}\left(\begin{array}{c}|(3\times 0.0)^4-(2\times 0.5)^4|+|(4\times 0.0)^4-(3\times 0.5)^4|+\\|(3\times 0.25)^4-(3\times 0.25)^4|+|(4\times 0.25)^4-(4\times 0.25)^4|+\\|(4\times 0.75)^4-(5\times 0.25)^4|+|(5\times 0.75)^4-(6\times 0.25)^4|\end{array}\right)\\+\frac{1}{3}\left(\begin{array}{c}|(1\times 0.4)^4-(2\times 0.0)^4|+|(2\times 0.4)^4-(3\times 0.0)^4|+\\|(2\times 0.3)^4-(2\times 0.67)^4|+|(3\times 0.3)^4-(3\times 0.67)^4|+\\|(3\times 0.3)^4-(4\times 0.33)^4|+|(4\times 0.3)^4-(5\times 0.33)^4|\end{array}\right)\\+\frac{1}{2}\left(\begin{array}{c}|(4\times 0.7)^4-(1\times 0.43)^4|+|(5\times 0.7)^4-(2\times 0.43)^4|+\\|(5\times 0.3)^4-(2\times 0.57)^4|+|(6\times 0.3)^4-(3\times 0.57)^4|\end{array}\right)\end{array}\right)$$
$$=34.924$$

4. The PULTSF ordered weighted distance operator

Inspired by the concepts of OWD (Ordered Weighted Distance) measures and the OWAD (Ordered Weighted Average Distance) operator, this paper constructs an ordered weighted distance measure for PULTSFSs. This approach not only emphasizes the importance of the ranking position of each deviation value but also provides a family of parameterized distance aggregation operators for measuring the distance between PULTSFSs.

Assume X is the set of all PULTSFNs, and let $\tilde{\boldsymbol{A}}(p)=\{\tilde{\alpha}_1(p),\tilde{\alpha}_2(p),...,\tilde{\alpha}_n(p)\}$ and $\tilde{\boldsymbol{B}}(p)=\{\tilde{\beta}_1(p),\tilde{\beta}_2(p),...,\tilde{\beta}_n(p)\}$ be two groups of PULTSFNs sets. Then, the PULTSF Weighted Distance

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(PULTSFWD) operator and the PULTSF Ordered Weighted Distance (PULTSFOWD) operator can be respectively defined as follows:

Definition 11. An n-dimensional PULTSF Weighted Distance operator is a mapping PULTSFWD: $\Omega^n \times \Omega^n \rightarrow R$, which has an associated weighting vector $w=(w_1,w_2,...,w_n)^T$, satisfying 0∈[0,1] and $\sum_{j=1}^{n} w_j = 1$. Its calculation formula is as follows:

$$PULTSFWD(\tilde{\boldsymbol{A}}(p),\tilde{\boldsymbol{B}}(p)) = \left(\sum_{j=1}^{n} w_j \left(D\left(\tilde{a}_j(p), \tilde{\beta}_j(p)\right)\right)^{\lambda}\right)^{1/\lambda}, \lambda > 0 \tag{10}$$

Definition 12. An n-dimensional PULTSF Ordered Weighted Distance operator is a mapping PULTSFOWD： $\Omega^n \times \Omega^n \rightarrow R$, which has an associated weighting vector $w=(w_1,w_2,...,w_n)^T$, satisfying 0∈[0,1] and $\sum_{j=1}^{n} w_j = 1$. Its calculation formula is as follows:

$$PULTSFOWD(\tilde{A}(p),\tilde{B}(p)) = \left(\sum_{j=1}^{n} w_j \left(D\left(\tilde{a}_{\sigma(j)}(p), \tilde{\beta}_{\sigma(j)}(p)\right)\right)^{\lambda}\right)^{1/\lambda}, \lambda > 0 \tag{11}$$

In the formula, $(\sigma(1), \sigma(2), \ldots, \sigma(n))$ is a permutation of (1,2,...,n) such that$D\left(\tilde{a}_{\sigma(j-1)}(p), \tilde{\beta}_{\sigma(j-1)}(p)\right) \geq D\left(\tilde{a}_{\sigma(j)}(p), \tilde{\beta}_{\sigma(j)}(p)\right)$.

The PULTSFOWD operator possesses the properties of commutativity, monotonicity, boundedness, idempotence, non-negativity, and reflexivity, but it does not always satisfy the triangle inequality. These properties can be established through the following theorems:

Theorem 2 (Commutativity --OWA aggregation). Suppose F is the PULTSFOWD operator, then

$$F\left(\left(\tilde{a}_1(p), \tilde{\beta}_1(p)\right), \cdots, \left(\tilde{a}_n(p), \tilde{\beta}_n(p)\right)\right) = F\left(\left(\dot{\tilde{a}}_1(p), \dot{\tilde{\beta}}_1(p)\right), \cdots, \left(\dot{\tilde{a}}_n(p), \dot{\tilde{\beta}}_n(p)\right)\right) \tag{12}$$

Where $\left(\left(\tilde{a}_1(p), \tilde{\beta}_1(p)\right), \cdots, \left(\tilde{a}_n(p), \tilde{\beta}_n(p)\right)\right)$ is any permutation of $\left(\left(\dot{\tilde{a}}_1(p), \dot{\tilde{\beta}}_1(p)\right), \cdots, \left(\dot{\tilde{a}}_n(p), \dot{\tilde{\beta}}_n(p)\right)\right)$

Proo： Let $F\left(\left(\tilde{a}_1(p), \tilde{\beta}_1(p)\right), \cdots, \left(\tilde{a}_n(p), \tilde{\beta}_n(p)\right)\right) = \left(\sum_{j=1}^{n} w_j \left(D\left(\tilde{a}_{\sigma(j)}(p), \tilde{\beta}_{\sigma(j)}(p)\right)\right)^{\lambda}\right)^{1/\lambda}$, $F\left(\left(\dot{\tilde{a}}_1(p), \dot{\tilde{\beta}}_1(p)\right), \cdots, \left(\dot{\tilde{a}}_n(p), \dot{\tilde{\beta}}_n(p)\right)\right) = \left(\sum_{j=1}^{n} w_j \left(D\left(\dot{\tilde{a}}_{\sigma(j)}(p), \dot{\tilde{\beta}}_{\sigma(j)}(p)\right)\right)^{\lambda}\right)^{1/\lambda}$.

Since $\left(\left(\tilde{a}_1(p), \tilde{\beta}_1(p)\right), \cdots, \left(\tilde{a}_n(p), \tilde{\beta}_n(p)\right)\right)$ is any permutation of $\left(\left(\dot{\tilde{a}}_1(p), \dot{\tilde{\beta}}_1(p)\right), \cdots, \left(\dot{\tilde{a}}_n(p), \dot{\tilde{\beta}}_n(p)\right)\right)$, it follows that if the expression $D\left(\tilde{a}_{\sigma(j)}(p), \tilde{\beta}_{\sigma(j)}(p)\right) = D\left(\dot{\tilde{a}}_{\sigma(j)}(p), \dot{\tilde{\beta}}_{\sigma(j)}(p)\right)$ holds for all j, then expression $F\left(\left(\tilde{a}_1(p), \tilde{\beta}_1(p)\right), \cdots, \left(\tilde{a}_n(p), \tilde{\beta}_n(p)\right)\right) = F\left(\left(\dot{\tilde{a}}_1(p), \dot{\tilde{\beta}}_1(p)\right), \cdots, \left(\dot{\tilde{a}}_n(p), \dot{\tilde{\beta}}_n(p)\right)\right)$ also holds.

Theorem 3 (Commutativity -- distance measure). Suppose F is the PULTSFOWD operator, then

$$F\left(\left(\tilde{a}_1(p), \tilde{\beta}_1(p)\right), \cdots, \left(\tilde{a}_n(p), \tilde{\beta}_n(p)\right)\right) = F\left(\left(\tilde{\beta}_1(p), \tilde{a}_1(p)\right), \cdots, \left(\tilde{\beta}_n(p), \tilde{a}_n(p)\right)\right) \tag{13}$$

Proo： Let $F\left(\left(\tilde{a}_1(p), \tilde{\beta}_1(p)\right), \cdots, \left(\tilde{a}_n(p), \tilde{\beta}_n(p)\right)\right) = \left(\sum_{j=1}^{n} w_j \left(D\left(\tilde{a}_{\sigma(j)}(p), \tilde{\beta}_{\sigma(j)}(p)\right)\right)^{\lambda}\right)^{1/\lambda}$, $F\left(\left(\tilde{\beta}_1(p), \tilde{a}_1(p)\right), \cdots, \left(\tilde{\beta}_n(p), \tilde{a}_n(p)\right)\right) = \left(\sum_{j=1}^{n} w_j \left(D\left(\tilde{\beta}_{\sigma(j)}(p), \tilde{a}_{\sigma(j)}(p)\right)\right)^{\lambda}\right)^{1/\lambda}$.

Because expression $D\left(\tilde{\alpha}_{\sigma(j)}(p),\tilde{\beta}_{\sigma(j)}(p)\right)=D\left(\tilde{\beta}_{\sigma(j)}(p),\tilde{\alpha}_{\sigma(j)}(p)\right)$ holds for all j, it follows that expression $F\left(\left(\tilde{\alpha}_1(p),\tilde{\beta}_1(p)\right),\cdots,\left(\tilde{\alpha}_n(p),\tilde{\beta}_n(p)\right)\right)=F\left(\left(\tilde{\beta}_1(p),\tilde{\alpha}_1(p)\right),\cdots,\left(\tilde{\beta}_n(p),\tilde{\alpha}_n(p)\right)\right)$ also holds.

Theorem 4 (Monotonicity). Let *F* be the PULTSFOWD operator, let $\widetilde{\boldsymbol{C}}(p)=\{\tilde{\gamma}_1(p),\tilde{\gamma}_2(p),\ldots,\tilde{\gamma}_n(p)\}$be a collection of PULTSFNs, If condition $D\left(\tilde{\alpha}_j(p),\tilde{\beta}_j(p)\right)\geq D\left(\tilde{\alpha}_j(p),\tilde{\gamma}_j(p)\right)$ holds for all *j*, then

$$F\left(\left(\tilde{\alpha}_1(p),\tilde{\beta}_1(p)\right),\cdots,\left(\tilde{\alpha}_n(p),\tilde{\beta}_n(p)\right)\right)\geq F\left(\left(\tilde{\alpha}_1(p),\tilde{\gamma}_1(p)\right),\cdots,\left(\tilde{\alpha}_n(p),\tilde{\gamma}_n(p)\right)\right) \tag{14}$$

Proof：Let $F\left(\left(\tilde{\alpha}_1(p),\tilde{\beta}_1(p)\right),\cdots,\left(\tilde{\alpha}_n(p),\tilde{\beta}_n(p)\right)\right)=\left(\sum_{j=1}^{n}w_j\left(D\left(\tilde{\alpha}_{\sigma(j)}(p),\tilde{\beta}_{\sigma(j)}(p)\right)\right)^{\lambda}\right)^{1/\lambda}$, $F\left(\left(\tilde{\alpha}_1(p),\tilde{\gamma}_1(p)\right),\cdots,\left(\tilde{\alpha}_n(p),\tilde{\gamma}_n(p)\right)\right)=\left(\sum_{j=1}^{n}w_j\left(D\left(\tilde{\alpha}_{\sigma(j)}(p),\tilde{\gamma}_{\sigma(j)}(p)\right)\right)^{\lambda}\right)^{1/\lambda}$.

Since for all *j*，$D\left(\tilde{\alpha}_j(p),\tilde{\beta}_j(p)\right)\geq D\left(\tilde{\alpha}_j(p),\tilde{\gamma}_j(p)\right)$，then for all *j*， expression $D\left(\tilde{\alpha}_{\sigma(j)}(p),\tilde{\beta}_{\sigma(j)}(p)\right)\geq D\left(\tilde{\alpha}_{\sigma(j)}(p),\tilde{\gamma}_{\sigma(j)}(p)\right)$holds, then $F\left(\left(\tilde{\alpha}_1(p),\tilde{\beta}_1(p)\right),\cdots,\left(\tilde{\alpha}_n(p),\tilde{\beta}_n(p)\right)\right)\geq F\left(\left(\tilde{\alpha}_1(p),\tilde{\gamma}_1(p)\right),\cdots,\left(\tilde{\alpha}_n(p),\tilde{\gamma}_n(p)\right)\right)$ also holds.

Theorem 5 (Boundary). Suppose *F* is the PULTSFOWD operator, then

$$\min_j\left\{D\left(\tilde{\alpha}_j(p),\tilde{\beta}_j(p)\right)\right\}\leq F\left(\left(\tilde{\alpha}_1(p),\tilde{\beta}_1(p)\right),\cdots,\left(\tilde{\alpha}_n(p),\tilde{\beta}_n(p)\right)\right)\leq\max_j\left\{D\left(\tilde{\alpha}_j(p),\tilde{\beta}_j(p)\right)\right\} \tag{15}$$

Proof：Let $\max_j\left\{D\left(\tilde{\alpha}_j(p),\tilde{\beta}_j(p)\right)\right\}=l$ and $\min_j\left\{D\left(\tilde{\alpha}_j(p),\tilde{\beta}_j(p)\right)\right\}=r$, and$\sum_{j=1}^{n}w_j=1$, then

$$F\left(\left(\tilde{\alpha}_1(p),\tilde{\beta}_1(p)\right),\cdots,\left(\tilde{\alpha}_n(p),\tilde{\beta}_n(p)\right)\right)=\left(\sum_{j=1}^{n}w_j\left(D\left(\tilde{\alpha}_{\sigma(j)}(p),\tilde{\beta}_{\sigma(j)}(p)\right)\right)^{\lambda}\right)^{1/\lambda}\leq\left(\sum_{j=1}^{n}w_jl^{\lambda}\right)^{1/\lambda}=l$$

$$F\left(\left(\tilde{\alpha}_1(p),\tilde{\beta}_1(p)\right),\cdots,\left(\tilde{\alpha}_n(p),\tilde{\beta}_n(p)\right)\right)=\left(\sum_{j=1}^{n}w_j\left(D\left(\tilde{\alpha}_{\sigma(j)}(p),\tilde{\beta}_{\sigma(j)}(p)\right)\right)^{\lambda}\right)^{1/\lambda}\geq\left(\sum_{j=1}^{n}w_jr^{\lambda}\right)^{1/\lambda}=r$$

Thus, expression $\min_j\left\{D\left(\tilde{\alpha}_j(p),\tilde{\beta}_j(p)\right)\right\}\leq F\left(\left(\tilde{\alpha}_1(p),\tilde{\beta}_1(p)\right),\cdots,\left(\tilde{\alpha}_n(p),\tilde{\beta}_n(p)\right)\right)\leq\max_j\left\{D\left(\tilde{\alpha}_j(p),\tilde{\beta}_j(p)\right)\right\}$ holds.

Theorem 6 (Idempotency). Suppose *F* is the PULTSFOWD operator. If for all *j*, it holds that $D\left(\tilde{\alpha}_j(p),\tilde{\beta}_j(p)\right)=d$，then

$$F\left(\left(\tilde{\alpha}_1(p),\tilde{\beta}_1(p)\right),\cdots,\left(\tilde{\alpha}_n(p),\tilde{\beta}_n(p)\right)\right)=d \tag{16}$$

Proof：Let $F\left(\left(\tilde{\alpha}_1(p),\tilde{\beta}_1(p)\right),\cdots,\left(\tilde{\alpha}_n(p),\tilde{\beta}_n(p)\right)\right)=\left(\sum_{j=1}^{n}w_j\left(D\left(\tilde{\alpha}_{\sigma(j)}(p),\tilde{\beta}_{\sigma(j)}(p)\right)\right)^{\lambda}\right)^{1/\lambda}$, Since for all *j*，$D\left(\tilde{\alpha}_j(p),\tilde{\beta}_j(p)\right)=d$，then expression $D\left(\tilde{\alpha}_{\sigma(j)}(p),\tilde{\beta}_{\sigma(j)}(p)\right)=d$ holds.Thus,

$$F\left(\left(\tilde{\alpha}_1(p),\tilde{\beta}_1(p)\right),\cdots,\left(\tilde{\alpha}_n(p),\tilde{\beta}_n(p)\right)\right)=\left(\sum_{j=1}^{n}w_j\left(D\left(\tilde{\alpha}_{\sigma(j)}(p),\tilde{\beta}_{\sigma(j)}(p)\right)\right)^{\lambda}\right)^{1/\lambda}=\left(\sum_{j=1}^{n}w_jd^{\lambda}\right)^{1/\lambda}=d$$

Thus, Theorem 6 is proved.

Theorem 7 (non-negativity). Suppose *F* is the PULTSFOWD operator, then

$$F\left(\left(\tilde{a}_1(p),\tilde{\beta}_1(p)\right),\cdots,\left(\tilde{a}_n(p),\tilde{\beta}_n(p)\right)\right)\geq 0 \quad (17)$$

The proof is straightforward and is omitted here.

Theorem 8 (Reflexivity). Suppose *F* is the PULTSFOWD operator, then

$$F\big((\tilde{a}_1(p),\tilde{\alpha}_1(p)),\cdots,(\tilde{a}_n(p),\tilde{\alpha}_n(p))\big)=0 \quad (18)$$

The proof is straightforward and is omitted here.

5. Families of the PULTSFOWD operator

By adjusting the weight vector *w* and the parameter λ, various specialized forms of the PULTSFOWD operator can be derived. The specific form chosen depends on the actual focus of the decision-maker in a given decision-making problem.

5.1 Analyzing the weighting vector

(1) When the weight vector satisfies w_1=1 and w_j=0 (∀*j*≠1), the PULTSF maximum distance is obtained;

(2) When the weight vector satisfies w_n=1 and w_j=0 (∀*j*≠*n*), the PULTSF minimum distance is obtained;

(3) When the weight vector satisfies w_k=1 and w_j=0 (∀*j*≠*k*), the step-type PULTSFOWD operator is obtained;

(4) When all weights are equal w_j=1/*n* for all *j*, the operator reduces to the PULTSF normalized distance (PULTFND);

(5) When the ranking positions of $D\big(\tilde{a}_j(p),\tilde{a}_j(p)\big)$ are consistent with those of $D\big(\tilde{a}_{\sigma(j)}(p),\tilde{a}_{\sigma(j)}(p)\big)$, the operator degenerates to the PULTSFWD operator.

5.2 Analyzing the parameter λ

(1) When λ=1, The PULTSFOWD operator reduces to (PULTSFOWHD) operator:

$$PULTSFOWHD(\tilde{A}(p),\tilde{B}(p))=\sum_{j=1}^{n}w_jD\big(\tilde{a}_{\sigma(j)}(p),\tilde{a}_{\sigma(j)}(p)\big) \quad (19)$$

In the formula, $(\sigma(1),\sigma(2),\ldots,\sigma(n))$ is a permutation of (1,2,...,*n*) such that $D\big(\tilde{a}_{\sigma(j-1)}(p),\tilde{a}_{\sigma(j-1)}(p)\big)\geq D\big(\tilde{a}_{\sigma(j)}(p),\tilde{a}_{\sigma(j)}(p)\big)$. Based on this, we further obtain:

(1.1) If all weights w_j=1/*n*, the PULTSF normalized Hamming distance is obtained;

(1.2)When the ranking position of $D\big(\tilde{a}_j(p),\tilde{a}_j(p)\big)$ coincides with that of $D\big(\tilde{a}_{\sigma(j)}(p),\tilde{a}_{\sigma(j)}(p)\big)$, the operator reduces to the PULTSFWHD operator.

(2) When λ=2, the PULTSFOWD operator reduces to the PULTSF Ordered Weighted Euclidean Distance (PULTSFOWED) operator:

$$PULTSFOWED(\tilde{\boldsymbol{A}}(p),\tilde{\boldsymbol{B}}(p))=\sqrt{\sum_{j=1}^{n}w_j\left(D\big(\tilde{a}_{\sigma(j)}(p),\tilde{a}_{\sigma(j)}(p)\big)\right)^2} \quad (20)$$

In the formula, $(\sigma(1),\sigma(2),\ldots,\sigma(n))$ is a permutation of (1,2,...,n), such that $D\big(\tilde{a}_{\sigma(j-1)}(p),\tilde{a}_{\sigma(j-1)}(p)\big)\geq D\big(\tilde{a}_{\sigma(j)}(p),\tilde{a}_{\sigma(j)}(p)\big)$. Based on this, we further obtain:

(2.1) If all weights w_j=1/*n*, the PULTSF normalized Euclidean distance is obtained;

(2.2)When the ranking position of $D\left(\tilde{a}_j(p), \tilde{a}_j(p)\right)$ coincides with that of $D\left(\tilde{a}_{\sigma(j)}(p), \tilde{a}_{\sigma(j)}(p)\right)$, the operator reduces to the PULTSFWED operator.

(3)When λ→0， The PULTSFOWD operator degenerates into the Probabilistic Uncertain Linguistic Term Set Fuzzy Ordered Weighted Geometric Distance (PULTSFOWGD) operator：

$$PULTSFOWGD(\widetilde{A}(p),\widetilde{B}(p)) = \prod_{j=1}^{n}\left(D\left(\tilde{a}_{\sigma(j)}(p), \tilde{a}_{\sigma(j)}(p)\right)\right)^{w_j} \tag{21}$$

In the formula, $(\sigma(1), \sigma(2), \ldots, \sigma(n))$ is a permutation of (1,2,…,n) such that$D\left(\tilde{a}_{\sigma(j-1)}(p), \tilde{a}_{\sigma(j-1)}(p)\right) \geq D\left(\tilde{a}_{\sigma(j)}(p), \tilde{a}_{\sigma(j)}(p)\right)$. Based on this, we further obtain:

(3.1) If all weights w_j=1/n， the PULTSF normalized geometric distance is obtained.

(3.2) When the ranking position of
$D\left(\tilde{a}_j(p), \tilde{a}_j(p)\right) is\ consistent\ with\ that\ of\ D\left(\tilde{a}_{\sigma(j)}(p), \tilde{a}_{\sigma(j)}(p)\right)$ the operator reduces to the PULTSFWGD operator.

6. The PULTSFOWD operator-based MCDM model

Let A={A1,A2,...,Am}be the set of alternatives, C={C1,C2,...,Cn}be the set of criteria (or attributes),with the corresponding weight vector $w=(w_1,w_2,\ldots,w_n)^T$, satisfying $w_j\in[0,1]$ and $\sum_{j=1}^{n} w_j = 1$. Experts use PULTSFNs to represent the evaluation information of each alternative under each criterion. These evaluation results are constructed into a decision matrix of PULTSF information. $\widetilde{\mathcal{D}} = \left[\tilde{d}_{ij}(p)\right]_{m\times n}$,$\tilde{d}_{ij}(p) = \left\langle\left\{\left[s_{\mu^L_{ij(t)}}, s_{\mu^U_{ij(t)}}\right]\left|\hat{p}_{ij(t)}\right.\right\}, \left\{\left[s_{\eta^L_{ij(r)}}, s_{\eta^U_{ij(r)}}\right]\left|\tilde{p}_{ij(r)}\right.\right\}\ \left\{\left[s_{\nu^L_{ij(w)}}, s_{\nu^U_{ij(w)}}\right]\left|\bar{p}_{ij(w)}\right.\right\}\right\rangle$ (i=1, 2,..., m; j=1, 2,..., n; t=1, 2,..., #T; r=1, 2,..., #R; w=1, 2,..., #W).

The MCDM proposed in the PULTSFS environment provides a systematic method for dealing with uncertain, hesitant and ambiguous complex information. The structure of the algorithm is designed to guide DMs from problem definition to the final ranking of alternatives. The whole process is described in detail as follows:

Step 1. Based on the initial PULTSF decision matrix $\widetilde{\mathcal{D}}$, the PULTSFNs are normalized by Definition 9, and then the cost-type and benefit-type criteria values are converted by Eq.(22) to realize the standardization of criteria evaluation information in the PULTSF environment. Furthermore, the standardized PULTSF decision matrix $\tilde{\mathcal{G}} = \left[\tilde{g}_{ij}(p)\right]_{m\times n}$ is obtained.

$$\tilde{g}_{ij}(p) = \begin{cases} \tilde{d}_{ij}(p) = \left(\left\{\begin{bmatrix} s_{\mu^L_{ij(t)}}, \\ s_{\mu^U_{ij(t)}} \end{bmatrix}\left|\hat{p}_{ij(t)}\right.\right\}, \left\{\begin{bmatrix} s_{\eta^L_{ij(r)}}, \\ s_{\eta^U_{ij(r)}} \end{bmatrix}\left|\tilde{p}_{ij(r)}\right.\right\}, \left\{\begin{bmatrix} s_{\nu^L_{ij(w)}}, \\ s_{\nu^U_{ij(w)}} \end{bmatrix}\left|\bar{p}_{ij(w)}\right.\right\}\right), j \in J_1 \\ \left(\tilde{d}_{ij}(p)\right)^c = \left(\left\{\begin{bmatrix} s_{\nu^L_{ij(w)}}, \\ s_{\nu^U_{ij(w)}} \end{bmatrix}\left|\bar{p}_{ij(w)}\right.\right\}, \left\{\begin{bmatrix} s_{\eta^L_{ij(r)}}, \\ s_{\eta^U_{ij(r)}} \end{bmatrix}\left|\tilde{p}_{ij(r)}\right.\right\}, \left\{\begin{bmatrix} s_{\mu^L_{ij(t)}}, \\ s_{\mu^U_{ij(t)}} \end{bmatrix}\left|\hat{p}_{ij(t)}\right.\right\}\right), j \in J_2 \end{cases} \tag{22}$$

where J_1 and J_2 denote utility and cost criteria, respectively.

Step 2. The PULTSF positive ideal solution (PIS) is determined from the PULTSF group decision matrix $\tilde{\mathcal{R}} = \left[\tilde{r}_{ij}(p)\right]_{m\times n}$. The PULTSF PIS is denoted as $\tilde{\mathcal{P}}_+ = \left\{\tilde{r}_1^+(p), \tilde{r}_2^+(p), \cdots, \tilde{r}_j^+(p), \cdots, \tilde{r}_n^+(p)\right\}$, where $\tilde{r}_j^+(p) = \left(\tilde{r}_{ij}(p)\left|\max_i\left\{Sc\left(\tilde{r}_{ij}(p)\right)\right\}\right.\right)$.

Step 3. The PULTSFOWD operator is used to calculate the distances between the ideal solution and each alternative. It should be noted that multiple variants of the PULTSFOWD operator can be adopted in this process.

Step 4. A decision is made based on the results obtained in Step 3. The alternatives are ranked in ascending order according to their result values, and the alternative with the smallest result value is selected as the optimal solution.

7. Illustrative example

The following example illustrates the application of the proposed method. With the advancement of China's ecological civilization concept and technological progress, as well as under the strategic goal of carbon peaking, an increasing number of enterprises have begun to incorporate green and environmental factors into their evaluation systems for cooperative partners. Company W has long been committed to national strategies and the practice of green development. After more than 40 years of continuous efforts, it has secured a leading position in the field of green construction for building materials. Currently, Company W intends to procure wall finishing coatings for an under-construction green technology demonstration building. After preliminary screening, four strong candidates, A1, A2, A3, and A4, have entered the final evaluation stage. It is assumed that enterprises with strong green competitiveness are selected based on the following five criteria: environmental harmlessness of products (C1), low-carbon energy use (C2), clean production processes (C3), waste resource utilization (C4), and enterprise sustainability (C5). All criteria are benefit-type. A panel of experts was invited and provided the weight vector for the five criteria as *w*=(0.23, 0.17, 0.22, 0.12, 0.26)$^{T.}$

Based on their professional knowledge and using a linguistic term set LTS $S_{[0,6]} = \{s_0, s_1, \cdots s_6\} = \{\text{very low, low, relatively low, medium, relativey high, high, very high}\}$, the expert team employed Probabilistic Uncertain Linguistic Term Set Fuzzy Sets (PULTSFS) to express evaluation information. The initial PULTSF evaluation matrix is shown in Table 1.

Table 1
Initial PULTSF information provided by experts

	C1	C2	C3	C4	C5
A1	$(\{[s_1,s_2]\vert 0.3,[s_2,s_3]\vert 0.7\}$, $\{[s_2,s_3]\vert 0.5,[s_3,s_4]\vert 0.5\}$, $\{[s_0,s_1]\vert 0.4,[s_1,s_2]\vert 0.6\})$	$(\{[s_0,s_1]\vert 0.6,[s_1,s_2]\vert 0.4\}$, $\{[s_1,s_2]\vert 0.6,[s_2,s_3]\vert 0.4\}$, $\{[s_1,s_2]\vert 0.5,[s_2,s_3]\vert 0.5\})$	$(\{[s_1,s_2]\vert 0.8,[s_2,s_3]\vert 0.2\}$, $\{[s_0,s_1]\vert 0.4,[s_1,s_2]\vert 0.6\}$, $\{[s_0,s_1]\vert 0.6,[s_1,s_2]\vert 0.4\})$	$(\{[s_1,s_2]\vert 0.6,[s_2,s_3]\vert 0.4\}$, $\{[s_1,s_2]\vert 0.5,[s_2,s_3]\vert 0.5\}$, $\{[s_2,s_3]\vert 0.3,[s_3,s_4]\vert 0.7\})$	$(\{[s_1,s_2]\vert 0.9,[s_2,s_3]\vert 0.1\}$, $\{[s_2,s_3]\vert 0.5,[s_3,s_4]\vert 0.5\}$, $\{[s_0,s_1]\vert 0.6,[s_1,s_2]\vert 0.4\})$
A2	$(\{s_3\vert 0.6,[s_3,s_4]\vert 0.4\}$, $\{[s_1,s_2]\vert 0.6,[s_2,s_3]\vert 0.4\}$, $\{[s_0,s_1]\vert 0.7,[s_1,s_2]\vert 0.3\})$	$(\{[s_1,s_2]\vert 0.3,[s_2,s_3]\vert 0.7\}$, $\{[s_1,s_2]\vert 0.5,[s_2,s_3]\vert 0.5\}$, $\{[s_1,s_2]\vert 0.6,[s_2,s_3]\vert 0.4\})$	$(\{[s_4,s_5]\vert 0.6,[s_5,s_6]\vert 0.4\}$, $\{[s_0,s_1]\vert 0.6,[s_1,s_2]\vert 0.4\}$, $\{[s_0,s_1]\vert 0.6,[s_1,s_2]\vert 0.4\})$	$(\{[s_2,s_3]\vert 0.3,[s_3,s_4]\vert 0.7\}$, $\{[s_0,s_1]\vert 0.6,[s_1,s_2]\vert 0.4\}$, $\{[s_0,s_1]\vert 0.8,[s_1,s_2]\vert 0.2\})$	$(\{[s_4,s_5]\vert 0.8,[s_5,s_6]\vert 0.2\}$, $\{[s_1,s_2]\vert 0.6,[s_2,s_3]\vert 0.4\}$, $\{[s_0,s_1]\vert 0.6,[s_1,s_2]\vert 0.4\})$
A3	$(\{[s_0,s_1]\vert 0.5,[s_1,s_2]\vert 0.5\}$, $\{[s_1,s_2]\vert 0.3,[s_2,s_3]\vert 0.7\}$, $\{[s_2,s_3]\vert 0.5,[s_3,s_4]\vert 0.5\})$	$(\{[s_3,s_4]\vert 0.4,[s_4,s_5]\vert 0.6\}$, $\{[s_2,s_3]\vert 1\}$, $\{[s_1,s_2]\vert 0.4,[s_2,s_3]\vert 0.6\})$	$(\{[s_2,s_3]\vert 0.7,[s_3,s_4]\vert 0.3\}$, $\{[s_1,s_2]\vert 0.5,[s_2,s_3]\vert 0.5\}$, $\{[s_0,s_1]\vert 0.4,[s_1,s_2]\vert 0.6\})$	$(\{[s_0,s_1]\vert 0.2,[s_1,s_2]\vert 0.8\}$, $\{[s_0,s_1]\vert 0.5,[s_1,s_2]\vert 0.5\}$, $\{[s_1,s_2]\vert 0.6,[s_2,s_3]\vert 0.4\})$	$(\{[s_4,s_5]\vert 0.4,s_5\vert 0.6\}$, $\{[s_1,s_2]\vert 0.5,[s_2,s_3]\vert 0.5\}$, $\{[s_1,s_2]\vert 0.6,[s_2,s_3]\vert 0.4\})$
A4	$(\{[s_1,s_3]\vert 0.6,[s_3,s_4]\vert 0.4\}$, $\{s_2\vert 0.6,[s_2,s_3]\vert 0.4\}$, $\{[s_0,s_1]\vert 0.7,[s_1,s_2]\vert 0.3\})$	$(\{[s_2,s_3]\vert 0.3,[s_3,s_4]\vert 0.7\}$, $\{[s_1,s_2]\vert 0.5,[s_2,s_3]\vert 0.5\}$, $\{[s_0,s_1]\vert 0.6,[s_1,s_2]\vert 0.4\})$	$(\{[s_4,s_5]\vert 0.5,[s_5,s_6]\vert 0.5\}$, $\{[s_2,s_3]\vert 0.4,[s_3,s_4]\vert 0.6\}$, $\{[s_1,s_2]\vert 0.2,[s_2,s_3]\vert 0.8\})$	$(\{[s_3,s_4]\vert 0.5,[s_4,s_5]\vert 0.5\}$, $\{[s_1,s_2]\vert 0.7,[s_2,s_3]\vert 0.3\}$, $\{[s_0,s_1]\vert 0.6,[s_1,s_2]\vert 0.4\})$	$(\{[s_0,s_1]\vert 0.5,[s_1,s_2]\vert 0.5\}$, $\{[s_0,s_1]\vert 0.7,[s_1,s_2]\vert 0.3\}$, $\{[s_1,s_2]\vert 0.4,[s_2,s_3]\vert 0.6\})$

7.1 Application

Step 1. Based on Definition 9, the PULTSFNs in Table 1 were normalized and standardized. Since all five criteria are benefit-type criteria, the data in Table 2 were obtained according to Formula (22).

Step 2. The positive ideal solution (PIS) is determined based on the comparison of score function values.

$$PIS = \begin{pmatrix} C_1: & \langle\{[s_3,s_3]|0.6,[s_3,s_4]|0.4\},\{[s_1,s_2]|0.6,[s_2,s_3]|0.4\},\{[s_0,s_1]|0.7,[s_1,s_2]|0.3\}\rangle \\ C_2: & \langle\{[s_3,s_4]|0.4,[s_4,s_5]|0.6\},\{[s_2,s_3]|0.5,[s_2,s_3]|0.5\},\{[s_1,s_2]|0.4,[s_2,s_3]|0.6\}\rangle \\ C_3: & \langle\{[s_4,s_5]|0.6,[s_5,s_6]|0.4\},\{[s_0,s_1]|0.6,[s_1,s_2]|0.4\},\{[s_0,s_1]|0.6,[s_1,s_2]|0.4\}\rangle \\ C_4: & \langle\{[s_3,s_4]|0.5,[s_4,s_5]|0.5\},\{[s_1,s_2]|0.7,[s_2,s_3]|0.3\},\{[s_0,s_1]|0.6,[s_1,s_2]|0.4\}\rangle \\ C_5: & \langle\{[s_4,s_5]|0.8,[s_5,s_6]|0.2\},\{[s_1,s_2]|0.6,[s_2,s_3]|0.4\},\{[s_0,s_1]|0.6,[s_1,s_2]|0.4\}\rangle \end{pmatrix}$$

Step 3. According to the constraint condition, q must be no less than 2.7, therefore, the initial value of q is set to 3.The PULTSFOWD operator and its variants are employed to calculate the distances between the ideal solution and each alternative, with the weight vector w=(0.23, 0.17, 0.22, 0.12, 0.26)^T， The computation results are shown in Table 3.

Step 4. The alternatives were ranked based on the principle of distance minimization, with the results presented in Table 4 and Figure 1 (where ">" denotes "superior to").

Table 2
Normalization and standardization of PULTSFNs

<table>
<tr><th></th><th>C1</th><th>C2</th><th>C3</th><th>C4</th><th>C5</th></tr>
<tr><td>A1</td><td>({[s_1,s_2]|0.3,[s_2,s_3]|0.7},
{[s_2,s_3]|0.5,[s_3,s_4]|0.5},
{[s_0,s_1]|0.4,[s_1,s_2]|0.6})</td><td>({[s_0,s_1]|0.6,[s_1,s_2]|0.4},
{[s_1,s_2]|0.6,[s_2,s_3]|0.4},
{[s_1,s_2]|0.5,[s_2,s_3]|0.5})</td><td>({[s_1,s_2]|0.8,[s_2,s_3]|0.2},
{[s_0,s_1]|0.4,[s_1,s_2]|0.6},
{[s_0,s_1]|0.6,[s_1,s_2]|0.4})</td><td>({[s_1,s_2]|0.6,[s_2,s_3]|0.4},
{[s_1,s_2]|0.5,[s_2,s_3]|0.5},
{[s_2,s_3]|0.3,[s_3,s_4]|0.7})</td><td>({[s_1,s_2]|0.9,[s_2,s_3]|0.1},
{[s_2,s_3]|0.5,[s_3,s_4]|0.5},
{[s_0,s_1]|0.6,[s_1,s_2]|0.4})</td></tr>
<tr><td>A2</td><td>({[s_3,s_3]|0.6,[s_3,s_4]|0.4},
{[s_1,s_2]|0.6,[s_2,s_3]|0.4},
{[s_0,s_1]|0.7,[s_1,s_2]|0.3})</td><td>({[s_1,s_2]|0.3,[s_2,s_3]|0.7},
{[s_1,s_2]|0.5,[s_2,s_3]|0.5},
{[s_1,s_2]|0.6,[s_2,s_3]|0.4})</td><td>({[s_4,s_5]|0.6,[s_5,s_6]|0.4},
{[s_0,s_1]|0.6,[s_1,s_2]|0.4},
{[s_0,s_1]|0.6,[s_1,s_2]|0.4})</td><td>({[s_2,s_3]|0.3,[s_3,s_4]|0.7},
{[s_0,s_1]|0.6,[s_1,s_2]|0.4},
{[s_0,s_1]|0.8,[s_1,s_2]|0.2})</td><td>({[s_4,s_5]|0.8,[s_5,s_6]|0.2},
{[s_1,s_2]|0.6,[s_2,s_3]|0.4},
{[s_0,s_1]|0.6,[s_1,s_2]|0.4})</td></tr>
<tr><td>A3</td><td>({[s_0,s_1]|0.5,[s_1,s_2]|0.5},
{[s_1,s_2]|0.3,[s_2,s_3]|0.7},
{[s_2,s_3]|0.5,[s_3,s_4]|0.5})</td><td>({[s_3,s_4]|0.4,[s_4,s_5]|0.6},
{[s_2,s_3]|0.5,[s_2,s_3]|0.5},
{[s_1,s_2]|0.4,[s_2,s_3]|0.6})</td><td>({[s_2,s_3]|0.7,[s_3,s_4]|0.3},
{[s_1,s_2]|0.5,[s_2,s_3]|0.5},
{[s_0,s_1]|0.4,[s_1,s_2]|0.6})</td><td>({[s_0,s_1]|0.2,[s_1,s_2]|0.8},
{[s_0,s_1]|0.5,[s_1,s_2]|0.5},
{[s_1,s_2]|0.6,[s_2,s_3]|0.4})</td><td>({[s_4,s_5]|0.4,[s_5,s_5]|0.6},
{[s_1,s_2]|0.5,[s_2,s_3]|0.5},
{[s_1,s_2]|0.6,[s_2,s_3]|0.4})</td></tr>
<tr><td>A4</td><td>({[s_1,s_3]|0.6,[s_3,s_4]|0.4},
{[s_2,s_2]|0.6,[s_2,s_3]|0.4},
{[s_0,s_1]|0.7,[s_1,s_2]|0.3})</td><td>({[s_2,s_3]|0.3,[s_3,s_4]|0.7},
{[s_1,s_2]|0.5,[s_2,s_3]|0.5},
{[s_0,s_1]|0.6,[s_1,s_2]|0.4})</td><td>({[s_4,s_5]|0.5,[s_5,s_6]|0.5},
{[s_2,s_3]|0.4,[s_3,s_4]|0.6},
{[s_1,s_2]|0.2,[s_2,s_3]|0.8})</td><td>({[s_3,s_4]|0.5,[s_4,s_5]|0.5},
{[s_1,s_2]|0.7,[s_2,s_3]|0.3},
{[s_0,s_1]|0.6,[s_1,s_2]|0.4})</td><td>({[s_0,s_1]|0.5,[s_1,s_2]|0.5},
{[s_0,s_1]|0.7,[s_1,s_2]|0.3},
{[s_1,s_2]|0.4,[s_2,s_3]|0.6})</td></tr>
</table>

Table 3
Aggregated results

	MAX	MIN	PULTSF	PULTSFOWHD	PULTSFOWED	PULTSFOWGD
A1	8.7055	2.596	5.2771	5.2628	5.6909	4.8152
A2	3.6948	0.0000	1.1066	1.1623	1.9273	0.0000
A3	11.8575	0.0000	4.6112	4.655	6.3078	0.0000
A4	9.099	0.0000	3.6335	3.7092	5.1937	0.0000

Table 4
Ordering of the strategies

Distance Operators	Ordering
MAX	A2>A1>A4>A3
MIN	A3=A2=A4>A1
PULTSF	A2>A4>A3>A1
PULTSFOWHD	A2>A4>A3>A1
PULTSFOWED	A2>A4>A1>A3
PULTSFOWGD	A3=A2=A4>A1

The results indicate that slight differences exist in the rankings of alternatives among different distance aggregation operators, and the decision outcome depends on the selected operator. However, when q=3, alternative A2 exhibits the smallest distance from the ideal solution across all operators, rendering it the optimal choice. In contrast, A1 consistently ranks poorly and lacks competitiveness in the decision-making process.

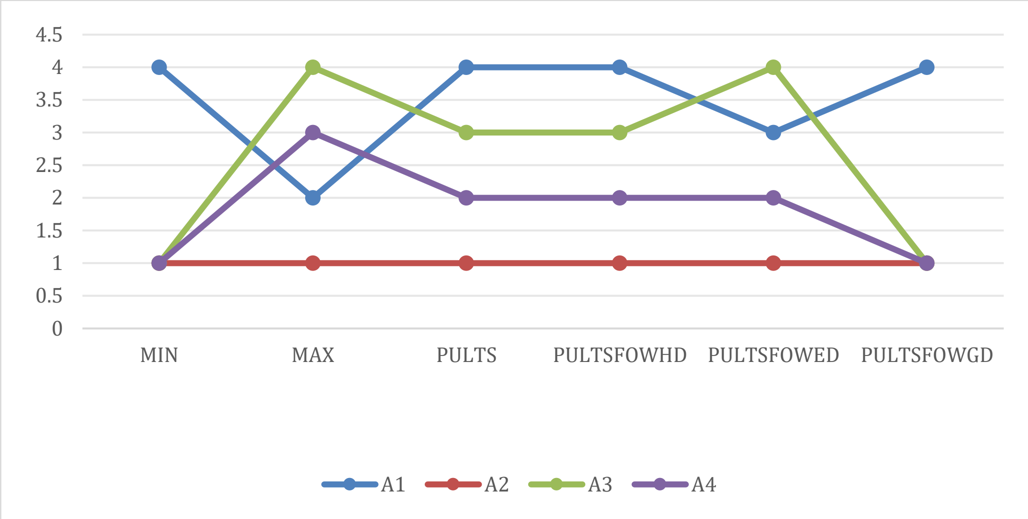

Fig. 1. Ranking Variations Across Strategies

7.2 Sensitivity analysis

To examine the impact of varying parameter q on the ranking outcomes of alternatives, the parameter *q* was assigned values within the range [3,7]. The corresponding results for each alternative, along with the rankings derived from the MAX, PULTSF, PULTSFOWHD, and PULTSFOWED operators, are presented in Table 5 and Figures 2-5.

Table 5
Results of alternative distance operators under varying parameter *q*

q		3	3.5	4	4.5	5	5.5	6	6.5	7
MAX	A1	8.7055	9.9663	16.4593	27.4257	46.0438	87.2204	141.4312	230.2231	384.8655
	A2	3.6948	22.355	42.1727	79.6996	150.9874	286.8553	546.6755	1045.198	2004.884
	A3	11.8575	7.4087	12.1473	19.9944	33.0383	81.697	133.6894	219.2871	360.6199
	A4	9.099	10.6715	17.9163	30.1971	51.0637	78.1597	132.292	225.0278	384.2686
MIN	A1	2.5957	3.6129	5.0199	6.9750	9.7005	13.5091	18.8424	26.3247	36.8392
	A2	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	A3	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	A4	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
PULTSF	A1	5.2771	7.4907	11.8924	19.0412	30.7426	55.7715	90.9148	149.1289	245.9908
	A2	1.1066	6.2139	11.2237	20.4675	37.635	87.0242	159.3824	293.8135	544.9856
	A3	4.6112	3.336	5.0266	7.6646	11.8227	23.8164	37.5378	59.5668	95.125
	A4	3.6335	4.9802	8.1314	13.4249	22.3741	20.2602	34.0728	57.7333	98.3999
PULTSFOWHD	A1	5.2628	7.3366	11.6366	18.5778	29.8879	53.9291	87.5417	143.4141	236.8297
	A2	1.1623	6.7614	12.2835	22.5142	41.5848	92.5265	170.5743	316.3894	590.2807
	A3	4.655	3.5026	5.3253	8.1551	12.6297	25.9606	41.0427	65.3181	104.5991
	A4	3.7092	4.992	8.1793	13.5428	22.6219	22.0272	37.0532	62.7785	106.9658
PULTSFOWED	A1	5.6909	7.7103	12.3775	20.0194	32.6501	60.7386	99.5414	164.4388	273.8245
	A2	1.9273	11.0743	20.7124	38.9106	73.4013	143.7638	271.5459	515.4549	983.0266
	A3	6.3078	4.5752	7.0971	11.1547	17.7874	40.8551	66.2647	107.9669	176.647
	A4	5.1937	6.8168	11.2862	18.8384	31.6531	38.4692	65.0812	110.6956	189.0742
PULTSFOWGD	A1	4.8152	6.8558	10.636	16.5471	25.8458	43.7061	68.5857	108.3802	172.1139
	A2	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	A3	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	A4	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

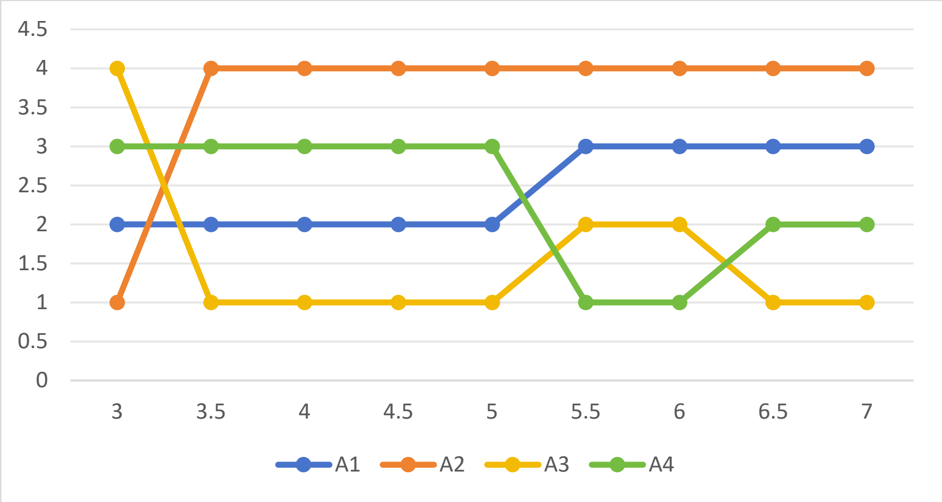

Fig. 2. Ranking variations under the MAX operator

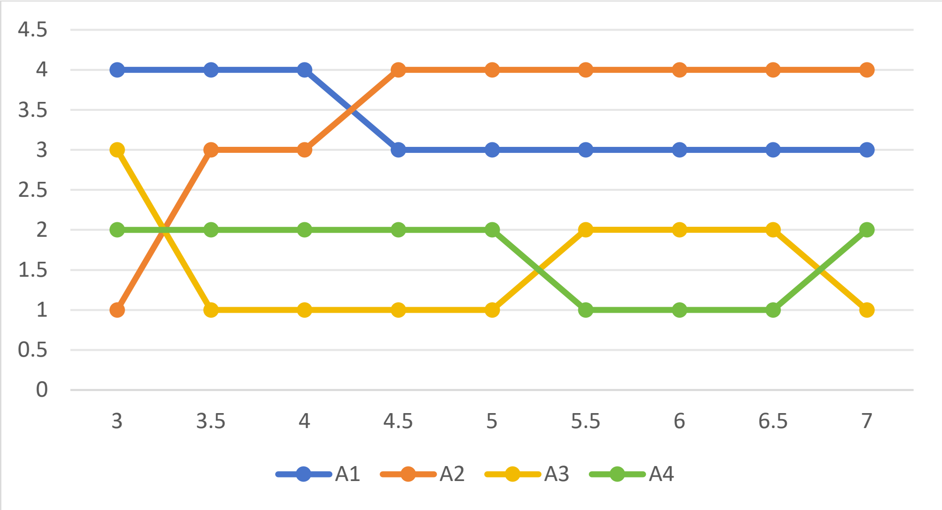

Fig. 3. Ranking variations under the PULTSF operator

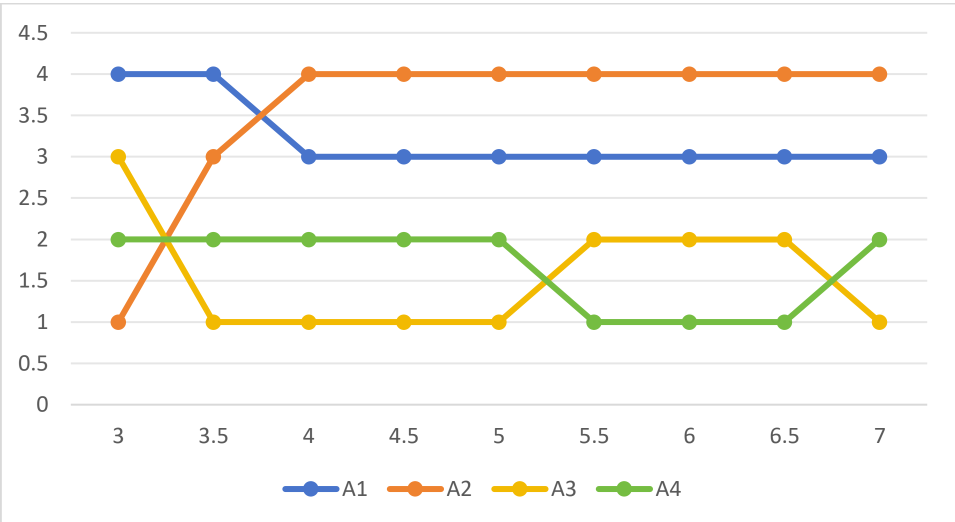

Fig. 4. Ranking variations under the PULTSFOWHD operator

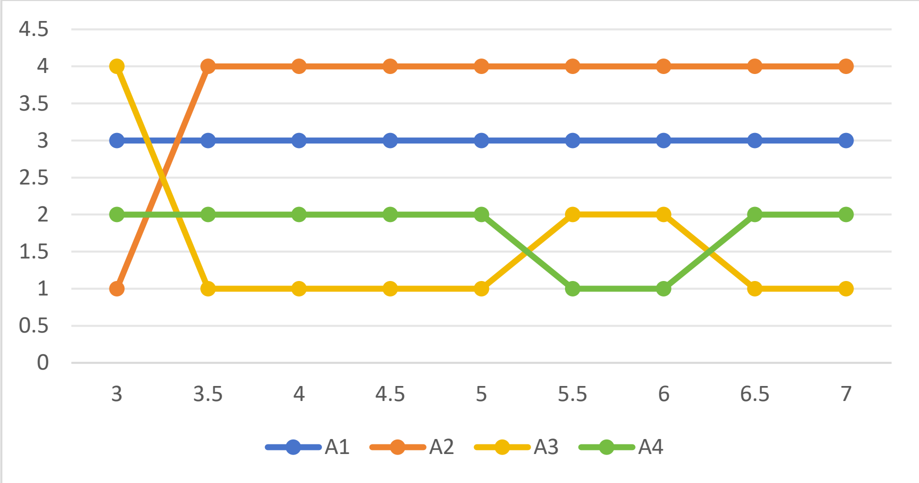

Fig. 5. Ranking variations under the PULTSFOWED operator

Under the MIN and PULTSFOWGD operators, alternative A1 consistently ranks last, performing the worst, while A2, A3, and A4 exhibit co-optimal characteristics and remain unaffected by fluctuations in the *q*-value. For the MAX, PULTSF, PULTSFOWHD, and PULTSFOWED distance operators, common "decision turning points" are observed around $q = 3.5$ and $q = 5.5$. When $q < 3.5$, A2 is always optimal; however, when $q > 4$, A2 becomes the worst-performing alternative. This indicates that the attribute distribution of A2 is only suitable for decision environments with low q-values. Near $q = 5.5$, fluctuations occur between A3 and A4, with both demonstrating comparable competitiveness under moderately high *q*-values. When q is sufficiently large (> 6.5), the ranking stabilizes across all distance operators as A3 > A4 > A1 > A2.

Compared to the MAX and PULTSF operators, the variations observed with the PULTSFOWHD and PULTSFOWED operators are smoother. In particular, the PULTSFOWHD operator maintains nearly parallel curves, and the ranking of A3 remains unaffected by changes in *q*. Thus, the PULTSFOWHD and PULTSFOWED operators exhibit lower sensitivity and higher robustness, making them more suitable for decision scenarios with high uncertainty in parameter settings.

8. Conclusion

This study validated the applicability of various distance aggregation operators in an intuitionistic fuzzy decision-making environment. Six operators—namely the maximum distance, minimum distance, PULTSF, PULTSFOWHD, PULTSFOWED, and PULTSFOWGD—were employed to calculate distances and rank four alternatives: A1, A2, A3, and A4. The results revealed minor differences in ranking outcomes across different operators, indicating that the final ranking is dependent on the selection of the aggregation operator. Variations in the weighting and measurement of deviation information among operators contributed to the divergence in ranking results, highlighting the flexibility of operator selection based on practical decision-making requirements.

Nevertheless, all distance operators consistently demonstrated that when $q = 3$, alternative A2 exhibited the smallest distance from the ideal solution, thereby being identified as the optimal choice, while alternative A1 consistently lacked competitiveness across all scenarios. As *q* increased, the ranking results stabilized across all operators, with A3 consistently occupying either the optimal or a co-optimal position. Sensitivity analysis further indicated that if the decision-making environment or the decision-maker's preference parameter *q* is uncertain, robust operators such as PULTSFOWHD or PULTSFOWED should be prioritized. Conversely, if preferences can be clearly defined and precisely adjusted via *q*, more sensitive operators such as MAX or PULTSF may be adopted.

The decision-making method proposed in this study provides an effective solution for multi-attribute decision-making problems under intuitionistic fuzzy environments. By introducing the IFOWD series of operators, the fuzziness and uncertainty inherent in decision information can be adequately addressed. Meanwhile, flexible adjustment of operator types allows for adaptation to different decision-making preferences, ensuring both scientific rigor and practical applicability. The findings of this research can be further extended to similar decision-making contexts such as investment decision-making, project evaluation, and resource allocation, offering a reference for optimal alternative selection in complex uncertain environments. Future research may focus on expanding the applicability of these operators, incorporating additional influencing factors to optimize the decision model, and exploring the integration of different types of intuitionistic fuzzy aggregation operators to enhance the comprehensiveness and precision of decision outcomes.

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Conflicts of Interest

The authors declare no conflicts of interest.

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